Enumerative Combinatorics M04

Problem Sheet 7

1. Find a bijection between binary rooted plane trees with n internal vertices and rooted plane trees with $n+1$ vertices.

2. Find a differential equation satisfied by the EGF $a(z)$ of the sequence $\{a_n\}_{n\geqslant 0}$ such that $a_{n+1} = a_n + n$ for all $n \geqslant 0$ and $a_0 = 1$. Knowing that the solution takes the form $\exp(z)P(z)$, where $P(z)$ is a polynomial, find $a(z)$.

3. Use the symbolic method to find the exponential generating function for the number of surjective maps from $[n]$ onto $[k]$ (for fixed k). Hence give an alternative proof of the formula found in problem 3 of Problem Sheet 2.

Note: All graphs in the next two problems are to be counted according to the number of vertices. The figure overleaf contains examples of the classes of graphs the problems deal with.

4. Given a labelled class $\mathcal{A}$, its *k-cycle* is the class $\mathcal{C}^k(\mathcal{A})$ defined as $\{(\alpha_1, \ldots, \alpha_k) : \alpha_i \in \mathcal{A}\}/\equiv$, where $\equiv$ denotes the following equivalence relation

$$(\alpha_1, \ldots, \alpha_k) \equiv (\beta_1, \ldots, \beta_k) \Leftrightarrow \alpha_i = \beta_{i+t} \text{ for some } t \text{ and all } i,$$

where the subindices are taken modulo k. For instance, if $\mathcal{A}$ is the class consisting only of a size-1 element, the elements in $\mathcal{C}^3(\mathcal{A})$ are $(1,2,3), (1,3,2)$. For a labelled class $\mathcal{A}$ with no size-0 objects, let $\mathcal{C}(\mathcal{A})$ be $\epsilon + \mathcal{A} + \mathcal{C}^2(\mathcal{A}) + \mathcal{C}^3(\mathcal{A}) + \cdots$ (you don't need to show it is indeed a class).

(i) Show that the EGF of $\mathcal{C}(\mathcal{A})$ is $\log\left(\frac{1}{1-a(z)}\right)$, where $a(z)$ is the EGF of the class $\mathcal{A}$.

(ii) A graph is said to be 2-*regular* if the degree of each vertex is exactly 2. Find the EGF for labelled 2-regular graphs. Check your result for the first initial values.

(iii) A graph is said to be *unicyclic* if it is connected and contains a unique cycle. Give the EGF for labelled unicyclic graphs in terms of the EGF for rooted labelled trees.

5. This problem shows how to view maps in terms of directed graphs. Roughly speaking, a directed graph (digraph for short) is like a normal graph but having arrows instead of edges (hence, the digraph with vertices

1 and 2 and an arrow from 1 to 2 has to be considered different from the digraph with an arrow from 2 to 1).

A map $f : [n] \to [n]$ can be represented as a digraph with vertex set $[n]$ with an arrow from x to $f(x)$, for all $x \in [n]$. This digraph is called the *functional graph* of f.

(i) Prove that the EGF for functional graphs is $\frac{1}{1-t(z)}$, where $t(z)$ is the EGF for labelled rooted trees. (You will probably need to use the previous problem.)

(ii) A map f is idempotent if $f^2 = f$. Describe the functional graph of an idempotent map and find the number of such maps.

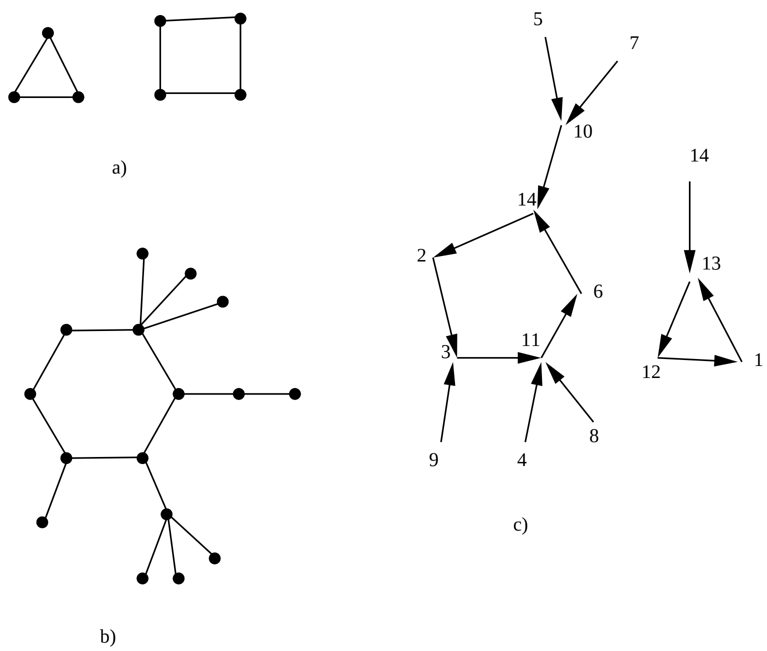

Figure 1: a)A 2-regular graph; b) A unicyclic graph; c) The functional graph of the map 13 3 11 11 10 14 10 11 3 14 6 1 12 13 , ie, $f(1) = 13, f(2) = 3$ and so on.