Enumerative Combinatorics M04

Problem Sheet 6

1. Find the generating function for the number of words of length n on the alphabet $[t]$ that do not contain k consecutive t's.

2. Use the symbolic method to obtain the generating function for the number of integer compositions with k parts. Deduce the well-known result that there are $\binom{n-1}{k-1}$ compositions of n with k parts. Find also the generating fuction for the number of compositions into k parts, with the parts being integers from a set $\Omega \in \mathbb{N}$.
Let us call a composition $n_1+n_2+\cdots+n_k$ *odd-even* if n_i has the same parity as i (ie, $3+2+1+6$ is an odd-even composition of 12, whereas $3+4+6+1$ is not). How does the number of odd-even compositions grow asymptotically? Which fraction of compositions of n are odd-even, for large n?

3. Consider words over an alphabet A with m letters. A *hidden pattern* of the word $p_1p_2\cdots p_r$ in a word $w_1\cdots w_n$ is a subsequence of letters w_{i_j} such that $w_{i_j} = p_j$ for all $1 \leq j \leq r$; ie, the following is a hidden pattern of the word "starter ": *Our**s** is essen**t**i**a**lly **a** tragic age, so we **r**efuse **t**o tak**e** it t**r**agically.*
Find the generating function for the number of hidden patterns of the word $p_1\cdots p_r$ in a word over the alphabet A. Which is the expected number of hidden pattern occurrences of $p_1\cdots p_r$ in a random word of length n?
Compare this to the expected number of contiguous ocurrences of $p_1\cdots p_k$; ie, ocurrences where the letters $p_1,\ldots,p_r$ appear in the same order and contiguous, as in *I'll have oysters as a **starter**.*
How realistic does this method seem to you for computing the number of hidden pattern ocurrences, say of the word "hello" in the course lecture notes?

4. Find the generating function for the number of paths from $(0,0)$ to $(n,0)$ using steps $U=(1,1)$, $D=(1,-1)$, and $H=(1,0)$. Find the coefficient of z^n in as simple form as you can. (You might want to find first the generating function for the number of such paths that do not cross the x-axis.)

5. All paths in this problem have steps $N=(0,1)$ and $E=(1,0)$. The number of paths from $(0,0)$ to (n,n) not containing any point below the line $x=y$ is trivially a Catalan number. Now consider paths from $(0,0)$ to $(2n,n)$ not containing any point below the line $x=2y$. Find the generating function for such paths. Can you guess and prove a formula for the generating function for the number of paths from $(0,0)$ to (kn,n) not containing any point below the line $x=kx$?

AdM