

Enumerative Combinatorics M04

Problem Sheet 5

- Let $\{a_n\}_{n \geq 0}$ be a sequence. Prove that the following are equivalent.
 - $\{a_n\}_{n \geq 0}$ satisfies a linear recurrence with constant coefficients $a_{n+k} + c_1 a_{n+k-1} + c_2 a_{n+k-2} + \cdots + c_k a_n = 0$.
 - Its generating function $a(z)$ is of the form $\frac{P(z)}{1+c_1z+c_2z^2+\cdots+c_kz^k}$, where $P(z)$ is a polynomial in z of degree at most $k-1$.
- Let b_n be the number of words of length n in the alphabet $\{a, b, c, d\}$ which contain an odd number of letters b . Show that $b_{n+1} = 4^n + 2b_n$ and find the corresponding generating function and an explicit formula for b_n . Show that $\{b_n\}_{n \geq 0}$ satisfies a linear recurrence with constant coefficients.
- Given two sequences $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$, its Hadamard product is the sequence $\{a_n b_n\}_{n \geq 0}$. Show that if $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$ have rational generating functions, then their Hadamard product also has a rational generating function.
- In how many ways can you give change of a pound (using 1,2,5,10, 20, 50, and 100 pence coins)?
(*Hint:* Find first the generating function $P(z)$ for the number of ways of giving change of n pence; then use the command `series(P(z), z, 101)` of MAPLE (or otherwise) to extract the coefficient you need.)
- Fix an integer $q \geq 1$. Show that for each $n \geq 1$, the number of partitions of n into parts that are not divisible by $q+1$ is equal to the number of partitions of n in which no part appears more than q times.
- An election is held with two candidates A and B ; each one receives n votes. Assuming all votes are put in the same box, in how many ways can we extract the votes one at a time, so that, at all times during the process, the number of votes we have counted for A is at least as many as the number of votes we have counted for B ?