Enumerative Combinatorics M04

Problem Sheet 4

1. Show that

(i) $x^{\bar{n}} = \sum_{k=1}^{n} \left[{n \atop k}\right] x^k$,

(ii) $x^n = \sum_{k=1}^{n} \left\{{n \atop k}\right\} x^{\underline{k}}$;

where $x^{\underline{k}} = x(x-1)\cdots(x-k+1)$ and $x^{\bar{n}} = x(x+1)\cdots(x+n-1)$.

Show also that $x^{\underline{n}} = \sum_{k=1}^{n} \left[{n \atop k}\right](-1)^{n+k} x^k$. Deduce that if $\{a_n\}$ and $\{b_n\}$ are sequences related by $b_n = \sum_{k=1}^{n} \left\{{n \atop k}\right\} a_k$ for all $n \geqslant 1$, then $a_n = \sum_{k=1}^{n} (-1)^{n+k} \left[{n \atop k}\right] b_k$.

2. Find the ordinary generating function of each of the following sequences in simple, closed form. (Each sequence is defined for $n \geqslant 0$.)

(i) $a_n = n$;

(ii) $a_n = \alpha n + \beta \quad (\alpha, \beta \in \mathbb{Q})$;

(iii) $a_n = n^2$;

(iv) $a_n = \binom{n}{k}$ (for fixed k);

(v) $a_n = \frac{n+1}{n!}$;

(vi) $a_n = 3^n$ if n is even and $a_n = 1$ if n is odd.

3. Find the OGF, in as simple form as possible, of the sequence $\{b_n\}_{n \geqslant 0}$, where b_n is the number of integers between 0 and $10^m - 1$ the sum of whose digits (in base 10) is n.

4. By setting up a recurrence and finding the corresponding generating function (or otherwise), find a closed formula for $a + 2a^2 + 3a^3 + 4a^4 + \cdots + na^n$.

5. Given a formal power series $f(z)$, we say that $g(z)$ is its functional inverse if $f(g(z)) = g(f(z)) = z$. Show that $f(z)$ has a functional inverse if and only if $[z^0]f(z) = 0$ and $[z]f(z) \neq 0$. Find the functional inverses of the series z, $\frac{1}{1-z} - 1$ and $\exp(z) - 1$.