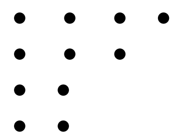
Enumerative Combinatorics M04

Problem Sheet 3

1. True or false? The Ferrer's diagram of the partition $(4, 3, 2, 1)$ is



2. How many solutions does the equation

$$x_1 + 2x_2 + 3x_3 + \cdots + nx_n = n \quad \text{with } x_i \geqslant 0$$

have?

How many solutions if we add the condition that $x_1 + x_2 + \cdots + x_n = k$?

3. Show that $p_n(2n) = p(n)$.

4. Fix $t \geqslant 0$. Show that as n tends to ∞, $p_{n-t}(n)$ becomes eventually constant. What is this constant $f(t)$? What is the least value of n for which $p_{n-t}(n) = f(t)$?

5. Prove that the number of maps f from $[n]$ to itself such that $f(1) = 1$ and $f(i) \leqslant 1 + \max\{f(j) : j < i\}$ is the Bell number $B(n)$.

6. A partition of the set $[n]$ is of type $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ if it contains α_i blocks of size i. Find a formula for the number of partitions of $[n]$ of type $(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

7. Show that

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum \binom{n}{n_1, n_2, \ldots, n_k},$$

where the sum is taken over all k-tuples $(n_1, n_2, \ldots, n_k)$ of strictly positive integers such that $n_1 + n_2 + \cdots + n_k = n$.

8. Suppose you are given a red box and n undistinguishable blue boxes. Prove that the number of ways of distributing the integers $\{1, \ldots, n\}$ into these boxes is $B(n + 1)$.

9. Which value of $(c_1, c_2, \ldots, c_n)$ maximizes the number of permutations in $\mathcal{S}_n$ of cycle type $(c_1, c_2, \ldots, c_n)$?

AdM