Enumerative Combinatorics M04

Problem Set 1

1.

(i) In how many ways can you fill a shelf if you have 51 different books but you can only fit 29 in the shelf? Suppose *Ulysses* is among your books. In how many ways can you fill the shelf if *Ulysses* is to be on it?

(ii) In how many ways can we arrange 17 boys and 13 girls in a row in such a way that there are no two consecutive girls?

(iii) How many different outcomes can we get if we throw 11 identical 6-faced dice?

(iv) In how many ways can 9 people sit in a round table? And if person 1 and person 2 do not want to sit together?

(v) How many maps from [10] to [20] are strictly increasing (ie, such that $f(i) < f(j)$ if $i < j$)?

(vi) Imagine that pence come in all possible coins from 1p to 30p. In how many ways can you pay at the coffee machine in the MI common room? (Note: coffee/tea is 30p.)

2. Prove the following identities of binomial numbers. Combinatorial proofs are encouraged!

(i) $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}, \quad n \geq r \geq k \geq 0.$

(ii) $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$

(iii) $\binom{m+n}{k} = \sum_{i=0}^{k}\binom{n}{i}\binom{m}{k-i}, \quad k \leq m+n.$

(iv) $\binom{n}{n} + \binom{n+1}{n} + \binom{n+2}{n} + \cdots + \binom{n+m}{n} = \binom{n+m+1}{n+1}.$

(v) $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$

3. How many subsets does $[n]$ have? How many pairs (A, B) are there such that $A \subseteq B \subseteq [n]$? How many t-tuples $(A_1, A_2, \ldots, A_t)$ are there such that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_t \subseteq [n]$?

4. Find the number of integer solutions of $x_1 + x_2 + \cdots + x_n = k$ such that $x_i \geq r \geq 0$ for all $1 \leq i \leq n$.

5. Find a bijection between the set of compositions of n whose first part is 1 and the set of compositions fo n whose first part is strictly greater than one. Deduce that the number of compositions of n with an even number of even parts is 2^{n-2}.

AdM