Arch. Hist. Exact Sci. (2009) 63:325–356
DOI 10.1007/s00407-009-0042-5

Euler's beta integral in Pietro Mengoli's works

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Received: 31 July 2008 / Published online: 27 March 2009

Abstract Beta integrals for several non-integer values of the exponents were calculated by Leonhard Euler in 1730, when he was trying to find the general term for the factorial function by means of an algebraic expression. Nevertheless, 70 years before, Pietro Mengoli (1626–1686) had computed such integrals for natural and half-integer exponents in his *Geometriae Speciosae Elementa* (1659) and *Circolo* (1672) and displayed the results in triangular tables. In particular, his new arithmetic–algebraic method allowed him to compute the quadrature of the circle. The aim of this article is to show how Mengoli calculated the values of these integrals as well as how he analysed the relation between these values and the exponents inside the integrals. This analysis provides new insights into Mengoli's view of his algorithmic computation of quadratures.

Introduction

The Beta Integral, known today as the Beta Function,[1]

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\mathrm{d}x, \quad p > 0,\ q > 0 \tag{1}$$

[1] Ferraro shows that Euler did not consider the Beta integral as a true function but only as an integral (Ferraro 2008).

Communicated by N. Guicciardini.

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became well known thanks to Euler (1707–1783), in the work *De progressionibus transcendentibus, seu quarum termini generales algebraice dari nequeunt*(1730).[2] Nevertheless, tables of values of these integrals had appeared 70 years before in two works by Pietro Mengoli (1626–1686)[3]: *Geometriae Speciosae Elementa* (1659) (referred to henceforth as *Geometriae*) and *Circolo* (1672). It is necessary to bear in mind that, 70 years prior to Euler's computations, the algebraic language was just beginning to take shape as a formal language, and neither Newton nor Leibniz had made algorithms for integration widely known.[4] Mengoli's calculations, like many of his other mathematical results, did not reach a wider public at that time, in part because of the complexity of his notation,[5] although they show the originality and the mathematical quality of this author (Massa-Esteve 1997, 2003, 2006a).

Mengoli's aim was the computation of the quadrature of the semicircle of diameter 1, which corresponds to the integral $\int_0^1 \sqrt{x(1-x)}\mathrm{d}x$, as we can read at the beginning of the *Circolo* (according to Mengoli the results of the *Circolo* date back to 1660),[6]

> I researched, since I was young, the problem of the quadrature of the circle, the most wanted problem in the Geometry…[7]

Instead of computing only the above quadrature,[8] Mengoli created a new arithmetic-algebraic method which involved the computation of countless quadratures.

In *Geometriae*, indeed, he introduced his method based on the construction of triangular tables and the use of the theory of quasi-proportions. He wanted to compute quadratures between 0 and 1 of geometric figures determined by $y = x^n(1-x)^{m-n}$, for natural numbers m and n. First, he displayed these figures in an infinite triangular

[2] Euler also dealt with Beta integral in other several works. For instance, for the treatment of infinite products one may consult Euler (1739, 1753). The name Beta function was assigned later in 1839 by Jacques Binet, see Cajori (1928–1929). For biographical information about Euler one may consult Gray (1985), Calinger (1996), Youschkevitch (1970–1991) and Massa-Esteve (2007).

[3] The name of Pietro Mengoli appears in the register of the University of Bologna in the period 1648–1686. He studied with Bonaventura Cavalieri and ultimately succeeded him in the chair of mechanics. He graduated in philosophy in 1650 and 3 years later in canon and civil law. He took holy orders in 1660 and was prior of the church of Santa Maria Maddalena in Bologna until his death. For more biographical information on Mengoli (see Natucci 1970–1991; Massa-Esteve 1998, 2006b; Baroncini and Cavazza 1986).

[4] It is unlikely that Euler was aware of Mengoli's construction algorithms. The approaches are very different since Euler's approach was based on analytical procedures. However, it is clear that Leibniz had had the opportunity in April 1676 to study Mengoli's *Circolo* (1672), and that he made extensive excerpts from this work (see Leibniz 1672–1676; Probst 2009). So one can consider the possibility that Euler could have known about Mengoli's work through Leibniz although we do not know of any evidence. On the other hand, Euler knew Wallis' works as we explain at the end of Sect. 2.2.

[5] In a letter to Collins, Isaac Barrow said that Mengoli's style was harder than Arabic and that if Mengoli had something new, he did not have the time to investigate it (Gregory 1939).

[6] In the opening pages of the *Circolo*, Mengoli explains that he had found this result, the quadrature of the circle, in 1660, but had not published it because, according to him, he only wanted to publish the mathematics he needed to explain natural events (Mengoli 1672). On squaring the circle one may consult Hobson (1913), Seidenberg (1981), Jami (1988) and Volkov (1997).

[7] "Cercai, fino da giovinetto, il Problema Della quadratura del Circolo, il più desiderato di tutti nella Geometria,..."(Mengoli 1672).

[8] In the seventeenth century squaring (quadrature) a geometric figure meant solving the problem of constructing a square of the area equal to the area of the given geometric figure.

table (*Tabula Formosa*), inspired by the combinatorial triangle (also known as Pascal's triangle).[9] He obtained an infinite triangular table of numerical values of their quadratures, which is nothing other than the harmonic triangle (also known as Leibniz's triangle).[10] Mengoli was thus able to compute these values, now known as Beta integrals, by means of binomial coefficients, expressed in modern notation as

$$\frac{1}{(m+1)\binom{m}{n}} = \int_0^1 x^n (1-x)^{m-n}\,\mathrm{d}x = B(n+1, m-n+1). \tag{2}$$

Subsequently in his *Circolo* (1672), Mengoli applied his method to compute infinitely many values of an interpolated table of quadratures. First, he displayed the figures $\int_0^1 \sqrt{x^n (1-x)^{(m-n)}}\mathrm{d}x$ for natural numbers m and n, in an infinite interpolated triangular table (*Interpolated Tabula Formosa*). Then he obtained an infinite interpolated triangular table of values of their quadratures, which is nothing other than the interpolated harmonic triangle. Mengoli was thus able to compute more values of Beta integrals, by means of terms of an interpolated combinatorial triangle (see Figs. 9, 10, where the notation with halves in the symbols is introduced), expressed in modern notation as:

$$\frac{1}{(m/2+1)\binom{m/2}{n/2}} = \int_0^1 \sqrt{x^n (1-x)^{(m-n)}}\mathrm{d}x = B\left(\frac{n}{2}+1, \frac{m-n}{2}+1\right). \tag{3}$$

With the help of the properties of combinatorial triangle, Mengoli was now able to fill the interpolated combinatorial triangle, except for an unknown number "a" which is closely related to the quadrature of the circle ($1/2a = \pi/8$). Mengoli obtained successive approximations of the number "a" in order to approximate the number π up to eleven decimal places.

The aim of this article is to explain this algorithmic computation and to show that Mengoli was clearly aware that there exist relations between the values of the quadratures of the curves determined by algebraic expressions and their exponents; that is, that Beta integrals $B(p, q)$ [in the identity (1)] can be expressed as functions of the numbers p, q, in the sense that they depend on p and q. The concept of a function as such did not exist in Mengoli's time. In fact, the notions of a function and of functional relations were introduced by Leibniz in 1673 and established in the eighteenth century by Euler in his fruitful works (Fraser 1989; Ferraro 2000; Delshams and Massa 2008).

[9] The combinatorial triangle has passed into history as Pascal's triangle because Blaise Pascal (1623–62) explained and proved the properties in a very clear style (see Bosmans 1924; Pascal 1954; Edwards 2002). Mengoli probably did not know of Pascal's treatise since it was published in 1665, but he might have well known its source, Hérigone's work. On Hérigone (see Massa-Esteve 2006c, 2008).

[10] The harmonic triangle, also called Leibniz's triangle, is formed by the reciprocal of the elements (binomial coefficients) of the binomial triangle times the row numbers (see Fig. 5). Its definition is related to the successive differences of the harmonic sequence (see Edwards 2002).

Mengoli, nevertheless, explicitly established relations of dependence between some sets of quadratures. To clarify this dependence, several properties and theorems, due to Mengoli, of the interpolated combinatorial and the harmonic triangle are accurately analysed in the present article.

1 The Beta integral: the expression of a quadrature by Pietro Mengoli

The Beta integral $B(n+1, m-n+1)$ corresponds to the quadrature of the geometric figure determined by the function $y = x^n(1-x)^{m-n}$ from 0 to 1, and the axis OX. In this section we will explain Mengoli's method of quadratures. Mengoli calculated these quadratures in a different way to that employed by mathematicians of his period (i.e. method of exhaustion, method of indivisibles and methods involving algebraic procedures), and, prior to the development of the algorithms of infinitesimal calculus.

1.1 Mengoli's proof of the quadratures of the figures determined by $y = x^n(1-x)^{m-n}$

While attempting in his *Geometriae* to find a sounder, more general methods for calculating quadratures,[11] Mengoli made an original use of algebraic tools for solving quadrature problems (Massa-Esteve 2006a). In his proofs, Mengoli brought together the three procedures mentioned above: Archimedes' method; Cavalieri's indivisibles, and the tools provided by Viète's *specious algebra*.[12]

Mengoli's new method of quadratures was based on the construction of triangular tables (inspired by the combinatorial triangle) of geometrical figures and the application of the theory of "quasi-proportions". By analogy with the combinatorial triangle, he constructed triangular tables of power sums, which enabled him to calculate the value of several summations of powers in the second book of the *Geometriae*.

In the third book of the *Geometriae*, Mengoli constructed his theory of "quasi-proportions", taking as a model the Euclidean theory of proportions, which enabled him to calculate the value of the limits of these summations (Massa-Esteve 1997).[13]

Mengoli based the theory of "quasi-proportions" on his new notion of the ratio "*quasi a number*". To introduce this concept, Mengoli considered values up to 10 in

[11] After showing that he was familiar with the method of indivisibles and could apply this method, Mengoli claimed that his purpose was to give solid foundations for a new method of calculating quadratures: "Meanwhile I left aside this addition that I had made to the Geometry of Indivisibles, because I was afraid of the authority of those who think false the hypothesis that the infinity of all the lines of a plane figure is the same as the plane figure. I did not publish it not because I agreed with them, but because I was doubtful of it, and I tried... to establish new and secure foundations for the same method of indivisibles or for other methods, which were equivalent." (Mengoli 1659).

[12] Indeed, Mengoli explained it at the beginning of this book: "Both geometries, the old form of Archimedes and the new form of indivisibles of my tutor, Bonaventura Cavalieri, as well as Viète's algebra, are regarded as pleasurable by the learned. Not through their confusion nor through their mixture, but through their perfect conjunction, a somewhat new form (of geometry will arise)-our own-which cannot displease anyone." (Mengoli 1659).

[13] Mengoli considered Euclid's *Elements* the book on mathematics *par excellence*. More information on his new theories of quasi proportions can be found in Massa-Esteve (1997) and Massa-Esteve (2003).

the ratio of the summation $\sum_{a=1}^{a=t-1} a$ to the power t^2. This ratio, which depends on the values of "t", is denoted by Mengoli *determinable indeterminate ratio* because it is determinate when we know the values of t.[14] For example, if $t = 3$ the ratio is 3 to 9; if $t = 4$ the ratio is 6 to 16, …; if $t = 10$ the ratio is 45 to 100, and Mengoli argues that this is nearer to 1/2 than any other ratio, and he uses the term ratio *quasi* 1/2 to denote the "limit" of this sequence of ratios.[15] The idea of ratio "*quasi a number*" suggests to us, albeit in a somewhat imprecise way, the modern concept of limit.[16] This notion, together with the idea of a determinable indeterminate ratio, was employed by Mengoli to establish the definitions of ratios "*quasi infinite*", "*quasi null*", "*quasi equality*" and "*quasi a number*". Thus, for example, he defined the ratio "quasi infinite" as

> 1. A determinable indeterminate ratio which, when determined, can be greater than any given ratio, as far as is thus determinable, will be called *quasi infinite*.[17]

And the ratio quasi equality thus

> 3. And one that can be smaller than any given ratio greater than equality; and greater than any given ratio smaller than equality, as far as it is thus determinable, will be called *quasi equality*. Or otherwise, that which can be nearer to equality, than any given ratio not equal to equality, as far as it is thus determinable, will be called *quasi equality*.[18]

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[14] By assigning different values to t, Mengoli explicitly introduces the concept of a "variable", a notion that was rather new at the time. There also seems to be certain dependence between the value of t and the value of the summation, yet while Mengoli implies an idea of sequence, his usage is still far from the general concept of function.

[15] Mengoli argues: "For different values of the letter t, ordered always in an increasing [sequence], there are different [ratios] and always ordered in increasing [sequence], but always smaller than the ratio 1/2; indeed, increasingly approaching 1/2 itself. That is, if the question could be propounded for any given value, I would answer that the ratio gets nearer to 1/2 than any other (given) ratio, [and] it would be called to the same indeterminate ratio $\sum_{a=1}^{a=t-1} a$ to t^2, quasi 1/2." (Mengoli 1659). Mengoli expressed the summation of the formula as the letter O and for expressing the exponents he wrote the number on the right side of the letter (Massa-Esteve 1997).

[16] In his *Circolo* (1672), Mengoli once again employed quasi ratios, and states: "Dissi quasi, e volsi dire, che vadino acostandosi ad essere precisamente tali." (Mengoli 1672).

[17] "1. Ratio indeterminata determinabilis, quae in determinari, potest esse maior, quam data, quaelibet, quatenus ita determinabilis, dicetur, Quasi infinita." (Mengoli 1659). To clarify the notion of "ratio quasi infinite" Mengoli considered values up to 10 in the ratio $O.a$ to t; for instance, if $t = 4$, then the ratio is 6 to 4; if $t = 7$ then the ratio is 21 to 7; if $t = 10$ then the ratio is 45 to 10. He argued that the ratio takes greater and greater values as the value of t increases, so the ratio is quasi infinite (Mengoli 1659).

[18] "3. Et quae potest esse minor, quàm data quaelibet minor inaequalitas; & maior, quàm data quaelibet minor inaequalitas, quatenus ita determinabilis, dicetur, Quasi aequalitas. Vel aliter, quae potest esse propior aequalitati, quàm data quaelibet non aequalitas, quatenus talis, dicetur, Quasi aequalitas."(Mengoli 1659). Mengoli also defines the ratio quasi null and the ratio quasi equal to a given ratio: "2. And one that can be smaller than any other given (ratio), as far as it is thus determinable, will be called *quasi null*. 4. And one that can be smaller than any ratio larger than a given ratio, and larger than any ratio smaller than the same given ratio, as far as it is thus determinable, will be called quasi equal to this given ratio. Or otherwise one that can be nearer to any given ratio than any other ratio not equal to it, as far as it is thus determinable, will be called quasi equal to the same (given) ratio." (Mengoli 1659).

With these definitions, Mengoli obtained ratios between many kinds of summations with $t-1$ addends and a power of the number t. He calculated what these ratios tend toward when t is very large, obtaining in particular that

$$(m+n+1)\binom{m+n}{n}\sum_{a=1}^{a=t-1} a^m\,(t-a)^n \tag{4}$$

tends to t^{m+n+1} when t tends to infinity, in the sense that their ratio can be made arbitrarily close to equality by making t sufficiently large.[19]

Other mathematicians of the period—such as Fermat, Roberval, Wallis, and Pascal—used different ways. They aimed, among other things, to calculate the result that today would be written $\lim \frac{1^p+\cdots+t^p}{t^{p+1}} = \frac{1}{p+1}$ for t tending to infinity, which correspond to the case $n=0$ in Mengoli's formula (4), in order to square the parabolas $y=x^n$, for n any positive integer.[20] It is obvious that Mengoli, like Roberval and Wallis, knew that the above limit was $\frac{1}{p+1}$. But the latter authors carried out the summations of powers and verified the resulting values only for a few cases of p. From these results they inferred the general rule and then applied it directly to the quadrature problem by taking limits of ratios between sums of ordinates and areas under curves. Instead Mengoli, after constructing the theory of quasi proportions to handle these limits, gave a proof that provided infinitely quadratures of the figures at once.

In the sixth book of his *Geometriae*, Mengoli defined his own system of co-ordinates[21] and described the geometric figures that he wanted to square as "extended by their ordinates". He denoted these geometric figures (which he referred to as forms)[22] by means of an algebraic expression written as $FO.a^n r^{m-n}.$, which in modern notation can be written as $\int_0^1 x^n(1-x)^{m-n}\mathrm{d}x$. In Mengoli's notation $FO.a^n r^{m-n}.$, "*FO.*" denotes the form (now we would call the integral of an expression from 0 to 1),

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[19] He based his proof on previous theorems that established that smaller powers could be ignored as t increases (see Massa-Esteve 1997).

[20] We may cite Gilles Personne de Roberval (1602–1675), who in 1636, in a letter written to Pierre de Fermat (1601–1665), enunciated the rule for finding the infinite sum of powers, and explained how he employed it for calculating quadratures. Fermat, for his part, stated in a letter to Cavalieri, written before 1644, that he had squared the parabolas, giving the rule and an example. Ten years later, Pascal arrived, apparently independently, at a similar conclusion in the work *Potestatum numericarum summa* (1654). In 1657, Fermat himself proved the quadratures for a positive rational number n. Furthermore, John Wallis (1616–1703) also proved these same quadratures in his *Arithmetica Infinitorum* (1655) using the sum of powers. Fermat was not published during his lifetime, his works circulating only in the form of letters and manuscripts. However, parts of his work appeared in other publications. For example, Hérigone's work contains an explanation of Fermat's method of tangents, and one may consult Hérigone (1642–1644) and Cifoletti (1990). For references to Fermat one may consult Fermat (1891–1922), Mahoney (1973), references to Roberval can be found in: Auger (1962), Walker (1986) to Wallis in: Wallis (1972) and on Pascal (1954).

[21] He defined the abscissa as our x, but in a segment measuring the unit u or t. Mengoli always worked within a finite base in which the abscissa was represented by the letter "a" and the remainder was represented by the letter "$r=t-a$" or "$1-a$", depending on whether the base was a given value t or the unit u (Massa-Esteve 2006a).

[22] The word figure or forma, which dates from the previous century, was identified by measuring the intensity of a given quality (Massa-Esteve 2006a).

$$FO.\ u.$$

$$FO.\ a. \qquad FO.\ r.$$

$$FO.\ a^2. \qquad FO.\ ar. \qquad FO.\ r^2.$$

$$FO.\ a^3. \qquad FO.\ a^2r. \qquad FO.\ ar^2. \qquad FO.\ r^3.$$

Fig. 1 *Tabula Formosa* (Mengoli 1659)

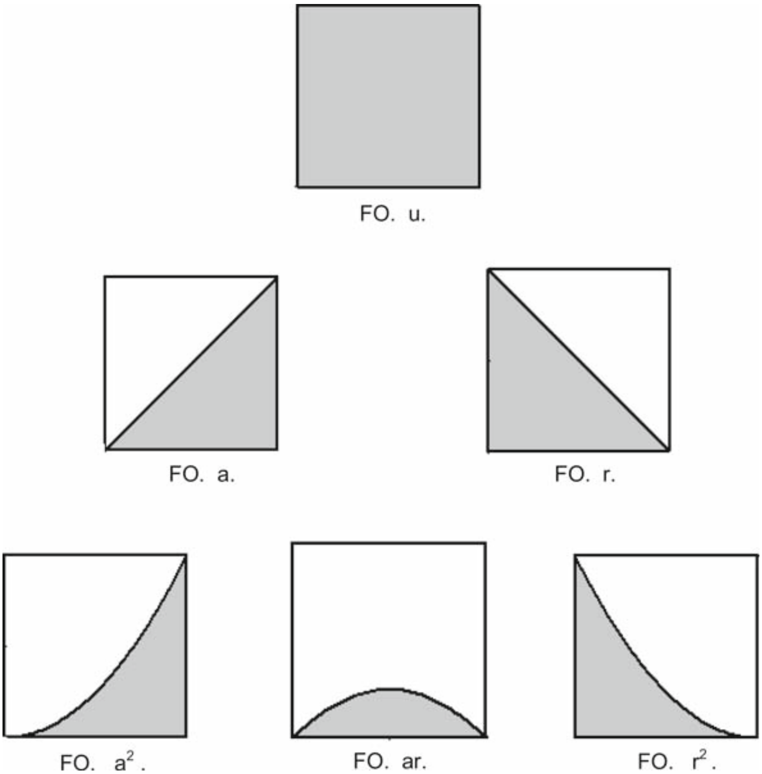


Fig. 2 The authors' sketches of geometric figures

a the abscissa (x) and r the remainder $(1 - x)$. He called this expression "Form of all products of n abscissae and $m - n$ remainders". In the singular case $m = n = 0$, Mengoli used $FO.\ u.$ ($\int_0^1 \mathrm{d}x$) and called this expression "Form of all rationals".

Mengoli went on to construct an infinite triangular table (called *Tabula Formosa*, Fig. 1).

The figure at the vertex represents a square of side 1; the two figures in the first row (called by Mengoli the "base of order one") represent two triangles; the three figures in the second row (the "base of order two") are determined by the ordinates of a parabola, and so on in the other rows (see Fig. 2).

We learn from the dedicatory letter of the sixth volume from his *Geometriae* that Mengoli had already computed the value of these quadratures by the method of

$$FO.\ u.$$
$$FO.\ 2a. \quad FO.\ 2r.$$
$$FO.3a^2. \ \ FO.6ar. \quad FO.\ 3r^2.$$
$$FO.\ \ a^3. \ \ FO.12a^2r. \ \ FO.12ar^2. \ \ FO.4r^3.$$

Fig. 3 *Tabula Quadraturarum* (Mengoli 1659)

indivisibles.[23] These values are related with the binomial coefficients. Indeed he multiplied each term $FO.a^n r^{m-n}$. of the *Tabula Formosa*, first by the binomial coefficient $\binom{m}{n}$ and second by the row number plus one unity $(m+1)$, obtaining a new table called *Tabula Quadraturarum* (Fig. 3) whose terms take simply the value 1.

In modern notation

$$(m+1)\binom{m}{n}\int_0^1 x^n(1-x)^{m-n}\mathrm{d}x = (m+1)\binom{m}{n} FO.a^n r^{m-n} = 1. \tag{5}$$

In order to prove that all terms of the *Tabula Quadraturarum* had value 1, Mengoli used the theory of quasi proportions, establishing ratios of quasi equality between the figures or forms. He first defined the "ascribed"[24], inscribed and circumscribed figures as being determined by rectangles built on the divisions of the base, working always with a finite number of divisions. Second, he proved that for any given ratio it is always possible to find a number of divisions of the base so that the ratio between the circumscribed and inscribed figures is nearer to equality than is the given ratio.

Finally, he took two ratios into account: one between the "ascribed" figure and the mixed-line figure whose quadrature he wished to find, and the other between the so-called "ascribed" figure and the square of side 1. He proved that for any figure in the table, when the number of divisions of the base was increased, these ratios were ratios of quasi equality. Then considering that for two given ratios of quasi equality with equal predecessors the successors would be equal, he proved in Proposition 10 that the square of side 1 was equal to any figure in the *Tabula Quadraturarum*; that is, that the areas of the geometric figures in the triangular table were of value 1.

It is noteworthy that for the first ratio Mengoli proved that as the number of divisions increases the ascribed figure is quasi equal to the mixed-line figure determined by the ordinates; that is to say, a geometric figure determined by rectangles approximates a mixed-line figure arbitrarily closely when the number of rectangles increases indefi-

23 The method of indivisibles was essentially made known with the publication of two works by Cavalieri: *Geometria indivisibilibus continuorum nova quadam ratione promota* (1635) and *Exercitationes geometricae sex* (1647). About the indivisibles of Cavalieri one may consult Giusti (1980), Andersen (1984/85), Massa-Esteve (1994) and Malet (1996).

24 The ascribed figure is determined by $t-1$ rectangles, when one divides the base in t parts. In fact the inscribed and ascribed figures are identical for any curve that is monotonically increasing. However, in general, the composite rectangles that make up the ascribed figure are sometimes smaller and sometimes larger than the inscribed figure. More information about these geometrical figures can be found in Massa-Esteve (2006a).

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

Fig. 4 Combinatorial triangle

nitely. To some extent, this first ratio of quasi equality recalls Archimedes' method of exhaustion.

For the second ratio Mengoli used algebraic procedures to establish, a proportion in which the first ratio is between a summation of powers and a power, and the second ratio is between a unit square and the ascribed figure. This proportion can be interpreted as equating a ratio between finite sums of ordinates to a ratio between figures and recalls to the method of indivisibles. The difference is that Mengoli applies to the proportion the theory of quasi proportions to find the ratio of quasi equality, and moreover, he does not establish proportions between infinite quantities since the ascribed figure is determined by a finite number of rectangles (Massa-Esteve 2006a).

1.2 Construction of the harmonic triangle

From equality (5) Mengoli deduced the relationship between the terms of *Tabula Formosa* and the values of the harmonic triangle. This relation would be deeply exploited by Mengoli in his subsequent work, the *Circolo*. So, in this section we describe how, starting from the combinatorial triangle, Mengoli constructed the harmonic triangle in *Circolo*.

Mengoli began by introducing the infinite combinatorial triangle $\binom{m}{n}$ (Fig. 4):

Considering the vertex separately, he multiplied each row by the number of its terms; that is, the first row by two, the second by three, and so on, thus the row m by $(m + 1)$, arriving at the table:

					1					
				2		2				
			3		6		3			
		4		12		12		4		
	5		20		30		20		5	
6		30		60		60		30		6

He then wrote the inverse $1/(m+1)\binom{m}{n}$ of these numbers in the form of a triangular table, obtaining what is nowadays known as the harmonic triangle (Fig. 5).

				1				
			1/2*		1/2			
		1/3		1/ 6		1/3		
	1/4		1/12		1/12		1/5	
1/5		1/20		1/30		1/20		1/5

Fig. 5 Harmonic triangle (Mengoli 1672). *Mengoli wrote 1(2) to express 1/2, 1(3) to express 1/3 and so on

Mengoli then identified these numbers with the values of the quadratures of the *Tabula Formosa*, as a consequence of the Proposition 10 (see previous section). In fact, he wrote down the harmonic triangle and below the *Tabula Formosa*, and asserted that their terms were homologous:[25]

> Which [figures] I have proved to be proportional to the quantities arranged in the third triangular table [harmonic triangle]; and the square of the Rational *FO.u*, is homologue to the unity; and the triangles *FO.a*, and *FO.r*, are homologues to the half; and the [parabolas] $FO.a^2$, $FO.ar$, $FO.r^2$, are homologues to a third, a sixth and a third of the unity; and $FO.a^3$, $FO.a^2.r$, $FO.a.r^2$, $FO.r^3$, are homologues to a fourth, a twelfth, a twelfth and a fourth of the unity, and of the square itself; and thus all the other forms in order, as also in the sixth element one may deduce by corollary from Proposition 10.[26]

This quotation is highly important, since it provides the algorithm for establishing the definition of the integrals from the *Tabula Formosa* according to their exponents. In other words, if the exponents of the algebraic expression associated to a squared mixed-line figure are known, a numerical value may then be assigned to it [see the identity (2)].

In Figs. 5, 6, 1 and 7, one may see the harmonic triangle, the *Tabula Formosa* in Mengolian notation and in modern notation, and the table of Beta integrals, respectively.

2 Mengoli's triangular table of Beta integrals B (p, q) with p and q rational

2.1 Mengoli's quadratures of the figures determined by $y = x^{n/2}(1-x)^{(m-n)/2}$

In this section, we will clarify Mengoli's procedure of construction of interpolated *Tabula Formosa*.

[25] The Mengolian homologous terms are discussed in the Sect. 3.4.

[26] "15. Le quali tutte hò dimostrato, che sono proportionali, come le quantità disposte nella terza tavola triangolare; ed è il quadrato della Rationale FO.u, homologa all'unità; e i triangoli FO. a, e FO. r, homologhi alle metà; e le FO. a2, FO. ar, FO. r2, homologhe alle parti dell'unità, terza, sesta, e terza; e le FO. a3, FO. a2r, FO. ar2, FO. r3, homologhe alle parti quarta, duodecima, duodecima, e quarta, dell'unità, e dello stesso quadrato; e così tutte le altre forme per ordine: come ivi nel sesto elemento si può dedurrè per corollario dalla prop. 10." (Mengoli 1672).

$$\int_0^1 1\,dx$$

$$\int_0^1 x\,dx \qquad \int_0^1 (1-x)\,dx$$

$$\int_0^1 x^2 dx \qquad \int_0^1 x(1-x)\,dx \qquad \int_0^1 (1-x)^2 dx$$

$$\int_0^1 x^3 dx \quad \int_0^1 x^2(1-x)\,dx \quad \int_0^1 x(1-x)^2 dx \quad \int_0^1 (1-x)^3 dx$$

$$\int_0^1 x^4 dx \quad \int_0^1 x^3(1-x)\,dx \quad \int_0^1 x^2(1-x)^2 dx \quad \int_0^1 x(1-x)^3 dx \quad \int_0^1 (1-x)^4 dx$$

Fig. 6 *Tabula Formosa* in modern notation

$$\mathrm{B}\,(1,1)$$

$$\mathrm{B}\,(2,1) \qquad \mathrm{B}\,(1,2)$$

$$\mathrm{B}\,(3,1) \quad \mathrm{B}\,(2,2) \quad \mathrm{B}\,(1,3)$$

$$\mathrm{B}\,(4,1) \quad \mathrm{B}\,(3,2) \quad \mathrm{B}\,(2,3) \quad \mathrm{B}\,(1,4)$$

$$\mathrm{B}\,(5,1) \quad \mathrm{B}\,(4,2) \quad \mathrm{B}\,(3,3) \quad \mathrm{B}\,(2,4) \quad \mathrm{B}\,(1,5)$$

Fig. 7 Table of Beta integrals

The *Tabula Formosa* is composed by geometric figures (see Fig. 1) determined by $y = x^n(1-x)^{m-n}$ with exponents m and n that are natural numbers. Mengoli extended the *Tabula Formosa* by interpolation to obtain quadratures of the geometric figures determined by $y = x^{n/2}(1-x)^{(m-n)/2}$, constructing a triangular table of values, what is nowadays known as Beta functions, $\mathrm{B}\,(p,q)$ with $p = n/2 + 1$ and $q = (m-n)/2 + 1, m, n \in \mathbb{N}$.

At the beginning of the *Circolo*, Mengoli states that he will make quadratures that no one has made before, and that he will make them by a method more intuitive than scientific:

> I therefore propose to clarify the quadrature of the circle and countless many quadratures that to my knowledge have never before been found or attempted by any geometer, with a procedure owing more to intuition [*intelligenza*] than to science, for in that way we have acquired our knowledge of numbers; and in exactly the same manner (*modo*), at once historical and theoretical, by which I have managed to find them, with evidence of the theory, without need of proof.[27]

[27] "Io propongo dunque la Quadratura del Circolo, ed altre innumerabili Quadrature, non mai, ch'io sappia, trovate, ò tentate da alcun Geometra, da dichiarare, per modo piu tosto d'intelligenza, come in noi stà la cognitione de' numeri, che di scienza; e nell'istesso modo à punto, col quale mi è riuscito di ritrovarle, historico, e dottrinale insieme, con l'evidenza della dottrina, senza bisogno di dimostrationi." (Mengoli 1672)

$$\begin{array}{c}
FO.\ u. \\
FO.\ a^{1/2}.\quad FO.\ r^{1/2}. \\
FO.\ a.\quad FO.\ (ar)^{1/2}.\quad FO.\ r. \\
FO.\ a^{3/2}.\ FO.\ (a^2r)^{1/2}.\ FO.\ (ar^2)^{1/2}.\ FO.\ r^{3/2}. \\
FO.\ a^2.\quad FO.\ (a^3r)^{1/2}.\ FO\ .ar.\quad FO.\ (ar^3)^{1/2}.\ FO.\ r^2. \\
FO.\ a^{5/2}.\ FO.\ (a^4r)^{1/2}.\ FO.\ (a^3r^2)^{1/2}.\ FO.\ (a^2r^3)^{1/2}.\ FO.\ (ar^4)^{1/2}.\ FO.\ r^{5/2}.
\end{array}$$

Fig. 8 Interpolated *Tabula Formosa* (Mengoli 1672)

Mengoli says in this quotation that his procedure is based on the (historical) method already employed by him in the *Geometriae* in 1659 (see the Sect. 1.1). He qualifies his method as "theoretical" since he is able to deduce the rule for the formation of the new interpolated table of quadratures. He also states that we human beings have intuitive knowledge of numbers, from which one may deduce that it is the basic intuition of the properties of the numbers, rather than science, that allows one to obtain infinitely many quadratures.

Mengoli began the construction of a new table of figures, interpolating the known geometric figures of *Tabula Formosa* for semi-integer exponents. He defined the new ordinates $[y = (x^n(1-x)^{m-n})^{1/2}]$ by geometric means

> "16. From these countless forms [geometric figures] one can conceive of other countless [in number] geometric means: that is from *FO. a.* $[y = x]$ another form can be made, in which those "orderly applied" [ordinates][28] are geometric means between the "tota"[29] and the abscissa, $[1{:}y = y{:}x]$, which we call roots of the abscissae : $[y = (x)^{1/2}]$."[30]

As he had done with the figures of the *Tabula Formosa*, he defined the forms corresponding to the roots, which he calls the "Form of all the roots of the abscissae", represented by *FO.R a*. For example, Mengoli wrote *FO.Rar.* to express $FO.(ar)^{1/2}.$;[31] in modern notation, $\int_0^1 x^{1/2}\,(1-x)^{1/2}\mathrm{d}x$. Once all the figures had been interpolated and defined, Mengoli placed them in an infinite triangular table (Fig. 8).

[28] The term "ordinata" is used in Mengoli's *Geometriae* (1659) instead of the word "applicata" which was commonly used at the time. Descartes defined the ordinates as "celles qui s'appliquents par ordre" (Descartes 1954). Mengoli, here, in *Circolo* (1672) named them "ordinatamente applicate". In the 1954 edition, there is the following editorial note: "The equivalent of 'ordination application' was used in the fifteenth century on translating Apollonius." The note also states that Hutton's *Mathematical Dictionary* of 1796 gave "applicata" as the word corresponding to the ordinate and explained that the expression "ordinata applicata" was also used. In fact Fermat and Cavalieri used "applicata".

[29] The term "tota" is defined by Mengoli as a straight line whose length is the unity.

[30] "16. Da queste innumerabili forme può farsi concetto d'altre innumerabili Geometricamente mezzane: cioè dalla *FO. a*, si può fare un'altra forma, nella quale le ordinatamente applicate, sono mezzane Geometriche trà la tota, e le abscisse, che si chiamano radici delle abscisse." (Mengoli 1672).

[31] For reader's convenience here and in the Fig. 12c we use 1/2 instead of the letter *R*.

A square is still represented at the vertex, in the first row, "$FO.a^{1/2}$." and "$FO.r^{1/2}$." are geometric figures determined by the curves $y = x^{1/2}$ and $y = (1-x)^{1/2}$, respectively; in the second row, "*FO. a.*", "*FO.*$(ar)^{1/2}$." and "*FO. r.*" are geometric figures determined by the curves $y = x$, $y = (x(1-x))^{1/2}$, which is the semi-circle and $y = 1 - x$, respectively, and so on. In this manner the table can be continued indefinitely.

Mengoli constructed the table of values of the quadratures associated with these interpolated figures by forming an interpolated harmonic triangle. To this end, he first constructed an interpolated combinatorial triangle from the combinatorial triangle. From this interpolated combinatorial triangle, and with the same procedure as for the natural exponents, he then constructed the interpolated harmonic triangle, with terms which in modern notation would be written thus:

$$\int_0^1 \sqrt{x^n (1-x)^{(m-n)}}\mathrm{d}x = \frac{1}{(m/2+1)\binom{m/2}{n/2}},$$

which indeed provides a definition of a binomial coefficient with half-integer coefficient. We show below the construction of this interpolated harmonic triangle.

2.2 Construction of the interpolated harmonic triangle

The combinatorial triangle (see Fig. 4) was known to most mathematicians of that time (Massa-Esteve 1997). However, Mengoli's originality stems from the properties (called *convenienzas* by Mengoli) used for the construction of the interpolated harmonic table. In order to understand the algorithm for constructing the values of terms of the interpolated *Tabula Formosa*, we first provide a description of the properties of the associated combinatorial triangle employed by Mengoli in his construction of the interpolated triangle, but in modern notation $\binom{m}{n}$.

Mengoli distinguished between sides (*latos*) and bases among the three kinds of lines that appear in the combinatorial triangle. These sides are the endless inclined lines of the above triangle, the first ones with a positive slope "/" , and the last ones with a negative slope "\". It is clear that in the combinatorial triangle there is a symmetry between the first sides and the last ones. Thus the first (and the last) side is formed by 1, 1, 1,... the second (and the penultimate) side is formed by the sequence of positive integers 1, 2, 3,... The third, fourth, fifth sides ... are formed by numbers called triangular, tetrahedral, pentagonal ... known since the Pythagoreans as *figurate numbers*. The bases are the horizontal lines, which we have previously called rows. Thus the first base is formed by 1, 1; the second base is formed by 1, 2, 1; the third one by 1, 3, 3, 1 and so on. Notice that the vertex, 1, is not included in any base. It is also worth noting that the three properties used by Mengoli in the construction of the interpolated harmonic triangle are based on the distribution of terms of the combinatorial triangle along sides (*latos*). In contrast, the two properties used by Mengoli to check this construction are only based on the distribution along rows (bases).

The **first property** of the sides employed by Mengoli in the construction affirms that each term on the first (and the last) side is obtained from the arithmetical mean of the preceding and succeeding terms; the terms on the first side $1, 1, 1, 1, \ldots$, as well as the terms on the second (and the penultimate) sides, $1, 2, 3, 4, 5, \ldots$ since $(1+3)/2 = 2$; $(2+4)/2 = 3$; $(3+5)/2 = 4\ldots$ This property is valid only for the first (and the last) two sides. Expressed in combinatorial numbers, it would be written, in modern notation, as:

$$\frac{\binom{m}{0}+\binom{m+2}{0}}{2}=\binom{m+1}{0}; \quad \frac{\binom{m}{1}+\binom{m+2}{1}}{2}=\binom{m+1}{1}.$$

The **second property** may be referred to as the property of figurate numbers. Thus, the triangular numbers on the third (and antepenultimate) side are obtained from the second (and penultimate) side by multiplying two consecutive terms and dividing by two; quoting Mengoli "the Triangular numbers: 1, 3, 6, 10, 15, 21, are the half of the product of 1 by 2, 2 by 3, 3 by 4, 4 by 5, 5 by 6, 6 by 7" (Mengoli 1672). In order to facilitate the reader's understanding, if we denote the terms on the second side by $a_1 = 1, a_2 = 2, a_3 = 3, \ldots, a_n = n$, and the triangular numbers (on the third side) by $a_1^t = 1, a_2^t = 3, a_3^t = 6, \ldots$ then the nth triangular number is given by $a_n^t = \frac{1}{2!} a_n . a_{n+1}$. For example: 1 = 1/2. (1.2); 3 = 1/2.(2.3); 6 = 1/2. (3.4); 10 = 1/2. (4.5); 15 = 1/2. (5.6), and so on.

Furthermore, as Mengoli also knew, the terms (on the fourth side) corresponding to the tetrahedral numbers are obtained by multiplying three consecutive terms from the second side and dividing by six. In an analogous way if we denote the tetrahedral numbers (on the fourth side) by $a_1^{\text{te}} = 1, a_2^{\text{te}} = 4, a_3^{\text{te}} = 10, \ldots$ then the nth tetrahedral number is given by $a_n^{\text{te}} = \frac{1}{3!} a_n . a_{n+1} . a_{n+2}$. For example: 1 = 1/6. (1.2.3); 4 = 1/6. (2.3.4); 10 = 1/6. (3.4.5); 20 = 1/6 (4.5.6)..., and so on. For the pentagonal numbers (on the fifth side) denoted by $a_1^p = 1, a_2^p = 5, a_3^p = 15, \ldots$ the nth pentagonal number is given by $a_n^p = \frac{1}{4!} a_n . a_{n+1} . a_{n+2} . a_{n+3}$, and so on.

Expressed in combinatorial numbers, the figurate numbers would be written, in modern notation, as:

$$\text{Triangular}: \binom{m+1}{2} = \frac{1}{2}\binom{m}{1}\binom{m+1}{1};$$

$$\text{Tetrahedral}: \binom{m+2}{3} = \frac{1}{6}\binom{m}{1}\binom{m+1}{1}\binom{m+2}{1};$$

$$\text{Pentagonal}: \binom{m+3}{4} = \frac{1}{24}\binom{m}{1}\binom{m+1}{1}\binom{m+2}{1}\binom{m+3}{1}.$$

The **third property** may be called the "property of the ratios", and is verified by all the sides of the table. On the second side, the terms 1, 2, 3, 4, 5, 6, ... have for ratios 1 : 2, 2 : 3, 3 : 4, 4 : 5, 5 : 6, ...; that is, each ratio is obtained from the preceding one by adding one unity to the antecedent and one unity to the consequent. We have skipped the first side since this property in this case is trivial. On the third side the terms 1, 3, 6, 10, 15, ... verify this property, since they have for ratios 1 : 3, 2 : 4, 3 : 5, ...; on the fourth side, the terms 1, 4, 10, 20, ... have for ratios 1 : 4, 2 : 5, 3 : 6, ... and analogously for the other sides. Expressed in combinatorial numbers and in modern notation, this relation would be written as:[32]

$$\frac{\binom{m}{n}}{\binom{m+1}{n}} = \frac{m-n+1}{m+1}.$$

Now we have described the three properties of the sides required by Mengoli for the construction of the interpolated triangle, we can begin to explain his procedure. He first wrote down the infinite harmonic triangle, or the table of values of the quadratures (in modern notation the values of $\int_0^1 x^n(1-x)^{m-n}\mathrm{d}x$), which he had already calculated.

To compute a new table of values of quadratures (in modern notation $\int_0^1 x^{n/2}(1-x)^{m-n/2}\,\mathrm{d}x$), Mengoli doubled the sides and bases of this table, putting asterisks in the places where the quadrature was unknown:

			1				
			*	*			
		1/2	*	1/2			
		*	*	*	*		
	1/3	*	1/6	*	1/3		
	*	*	*	*	*	*	
1/4	*	1/12	*	1/12	*	1/4	

Mengoli recalled that for the construction of the harmonic triangle he had used the combinatorial triangle, in which he had multiplied each row by its order plus one and then proceeded to insert the inverse values. In this way, by working backwards he arrived at the following tables:

[32] It is worth noting that with the first and the second property all terms of the combinatorial triangle are determined. Analogously these terms remain determined with the first and the third property. In contrast all three properties will be necessary for determining the terms of the interpolated harmonic triangle.

1
* *
2 * 2
* * * *
3 * 6 * 3
* * * * * *
4 * 12 * 12 * 4

1
* *
1 * 1
* * * *
1 * 2 * 1
* * * * * *
1 * 3 * 3 * 1

Mengoli then began filling in the empty spaces of the first (and the last) and the third (and the antepenultimate) sides; in fact, we might say that he was applying the first property of the combinatorial triangle. The first (and the last) side is formed by ones[33] and for each term with an asterisk of the third (and the antepenultimate) side he inserted the arithmetical mean of the preceding and the succeeding terms according to the first property.[34] Thus the value of the term between 1 and 2 is 3/2, and the value between 2 and 3 is 5/2, and so on.

1
1 1
1 * 1
1 3/2 3/2 1
1 * 2 * 1
1 * 5/2 5/2 * 1
1 * 3 * 3 * 1
1 * 7/2 * * 7/2 * 1

[33] "Now we begin to complete the empty spaces in this table. And first of all it is necessary to fill in the sides of this last table with unities" (Mengoli 1672).

[34] "Furthermore, it is necessary for the vacant spaces on third and antepenultimate sides, in which one sees all the numbers placed alternately from the unity, to be filled with the terms obtained with the arithmetic mean, so that the terms on all the third side and on all the antepenultimate side are ordered arithmetically from the unity, continually adding one half at a time" (Mengoli 1672).

Mengoli then considered the sides of odd order that had yet to be filled in, that is, the fifth, the seventh, etc., and employed now the second property of the combinatorial triangle (the property of figurate or polygonal numbers). The terms on the fifth side are obtained from the third side by multiplying two consecutive spaced out (*intervallatamente*) terms and dividing by two, according to the second property of the combinatorial triangle: 1, *, 3, *, 6,… are the half of the product of 1 by 2, 3/2 by 5/2, 2 by 3, 5/2 by 7/2, 3 by 4, and so on. Thus, for the first asterisk on the fifth side take the value $15/8 = 1/2 \cdot 3/2 \cdot 5/2$, for the second one take $35/8 = 1/2 \cdot 5/2 \cdot 7/2$, and so on.[35] Analogously, the terms on the seventh side are also obtained from the third side by multiplying three consecutives terms and dividing by six, according to the second property: 1, *, 4, *, 10, … are the sixth part of the product of 1 by 2 by 3, 3/2 by 5/2 by 7/2, 2 by 3 by 4,… and so on. Thus the first asterisk on the seventh side takes the value $105/48 = 1/6 \cdot 3/2 \cdot 5/2 \cdot 7/2$, and so on. [36]

1

1 1

1 * 1

1 3/2 3/2 1

1 * 2 * 1

1 15/8 5/2 5/2 15/8 1

1 * 3 * 3 * 1

1 105/48 7/2 35/8 35/8 7/2 105/48 1

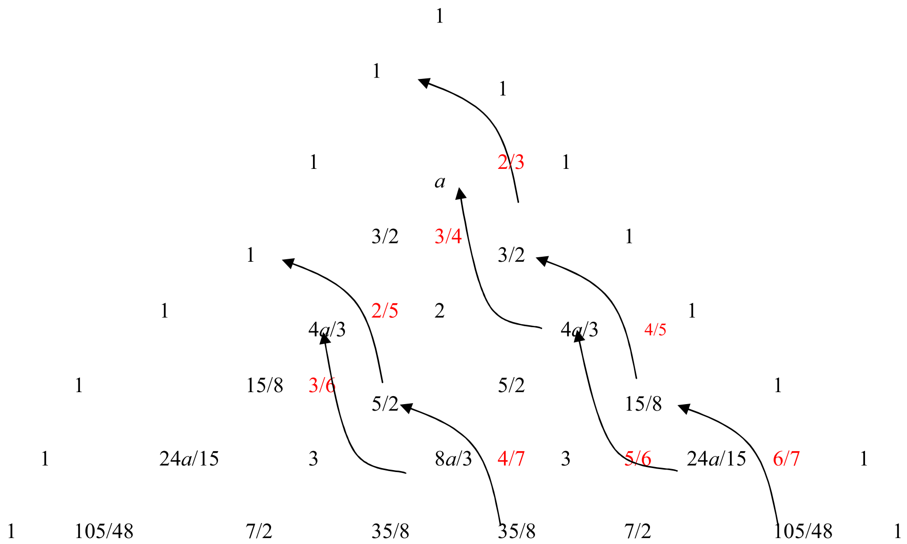
Mengoli then proceeded to calculate the asterisks in the sides of even order, that is, the second side, the fourth, the sixth, etc.. The terms on the second side are 1, *, 3/2, *, 15/8, *, 105/48, … have for ratios 2:3, *:*, 4:5, *:*, 6:7, … That is, the ratio of the first term to the third term is 2:3 (1:3/2), the ratio of the third term to the fifth term is 4:5 (3/2:15/8) and so on. Applying the third property, the ratio of the second term to the fourth term will be 3: 4, the ratio of the fourth term to the sixth term will then be 5:6, and so successively the other ratios are obtained. Mengoli stated that it is necessary to situate the letter a, the *"specious" number*, as the second term in the second side,

35 "In a like manner, since on the fifth side, and on the fifth from the end, one sees inserted the same numbers as on the third and antepenultimate sides in the second table [table of combinatorial numbers], which are the Triangular numbers: 1, 3, 6, 10, 15, 21, half of the product of 1 by 2, 2 by 3, 3 by 4, 4 by 5, 5 by 6, 6 by 7, situated alternately on the third and antepenultimate sides of this last table, and situated on the second and penultimate sides of the second table successively; I have judged it necessary to fill in the empty spaces on the fifth and the fifth from the end of this last table, with the halves of the products of 3/2 by 5/2, 5/2 by 7/2, 7/2 by 9/2 and so on; that is, 15/8, 35/8, 63/8 and so on" (Mengoli 1672).

36 Mengoli in fact explained how to fill in the ninth side, with the twenty-fourth part of the product of the four consecutive spaced out numbers from the third side, and the eleventh side with the 120th part of the product of the five numbers.

which enables him to fill in the vacant spaces;[37] for example, starting from 2:3, the value of the following ratio must be 3:4, then for the fourth term, he inserted the value $4a/3$ since $\frac{a}{*} = \frac{2+1}{3+1} = \frac{3}{4}$.

For a better understanding we sketch the procedure followed by Mengoli in the table below.



Finally, the interpolated combinatorial triangle appears as in Fig. 9.

Here the specious number a still has to be determined.[38] We can express this interpolated combinatorial triangle in terms of binomial coefficient containing semi-integer indices. In modern notation, the triangle is Fig. 10.

Finally, in order to construct the interpolated *Tabula Formosa*,[39] Mengoli used the same reasoning as that for the construction of the *Tabula Formosa*

[37] "28. It still remains to fill in the sides of even order; they are the even spaces on the even rows [second, fourth, sixth,...], counting from below the vertex. And since on the second side, the first term to the third is 2 to 3; the third to the fifth, 4 to 5; the fifth to the seventh, 6 to 7: it seems to us fitting that the second to the fourth should be 3 to 4; the fourth to the sixth, 5 to 6; the sixth to the eighth, 7 to 8. And provisionally situating a in the second place on the second and penultimate sides; that is, the second and the penultimate on the second base [row] below the vertex, we obtain the terms for the fourth space on the second and penultimate sides, which are the second and the penultimate on the fourth base [row], $4a/3$: they are the sixth place terms on the second and penultimate sides, which are the second and penultimate on the sixth base [row] $24a/15$. And thus to infinity, with the letter a, the "*specious" number*, all the vacant spaces on the second and penultimate sides will be provisionally filled, and the second and penultimate of all the rows in the triangular Table, as can be seen here" (Mengoli 1672).

[38] In the 17th century "specious number" meant a letter or a symbol that represented any magnitude, whether discrete or continuous. On Mengoli's "specious language" (see Massa-Esteve 2006a).

[39] Before transforming this interpolated combinatorial triangle (see Fig. 9) to the interpolated harmonic triangle (see Fig. 11) Mengoli checked the values of its terms by using two more properties of its bases.

$$
\begin{array}{c}
1 \\
1 \quad\quad 1 \\
1 \quad\quad a \quad\quad 1 \\
1 \quad\quad 3/2 \quad\quad 3/2 \quad\quad 1 \\
1 \quad\quad 4a/3 \quad\quad 2 \quad\quad 4a/3 \quad\quad 1 \\
1 \quad\quad 15/8 \quad\quad 5/2 \quad\quad 5/2 \quad\quad 15/8 \quad\quad 1 \\
1 \quad\quad 24a/15 \quad\quad 3 \quad\quad 8a/3 \quad\quad 3 \quad\quad 24a/15 \quad\quad 1 \\
1 \quad\quad 105/48 \quad\quad 7/2 \quad\quad 35/8 \quad\quad 35/8 \quad\quad 7/2 \quad\quad 105/48 \quad\quad 1
\end{array}
$$

Fig. 9 Interpolated combinatorial triangle (Mengoli 1672)

$$
\begin{array}{c}
\binom{0}{0} \\
\binom{1/2}{0} \quad \binom{1/2}{1/2} \\
\binom{1}{0} \quad \binom{1}{1/2} \quad \binom{1}{1} \\
\binom{3/2}{0} \quad \binom{3/2}{1/2} \quad \binom{3/2}{1} \quad \binom{3/2}{3/2} \\
\binom{2}{0} \quad \binom{2}{1/2} \quad \binom{2}{1} \quad \binom{2}{3/2} \quad \binom{2}{2} \\
\binom{5/2}{0} \quad \binom{5/2}{1/2} \quad \binom{5/2}{1} \quad \binom{5/2}{3/2} \quad \binom{5/2}{2} \quad \binom{5/2}{5/2} \\
\binom{3}{0} \quad \binom{3}{1/2} \quad \binom{3}{1} \quad \binom{3}{3/2} \quad \binom{3}{2} \quad \binom{3}{5/2} \quad \binom{3}{3} \\
\binom{7/2}{0} \quad \binom{7/2}{1/2} \quad \binom{7/2}{1} \quad \binom{7/2}{3/2} \quad \binom{7/2}{2} \quad \binom{7/2}{5/2} \quad \binom{7/2}{3} \quad \binom{7/2}{1/2}
\end{array}
$$

Fig. 10 Interpolated combinatorial triangle expressed with binomial coefficient

In the same manner, one constructs a new table, multiplying the term of the vertex by 1, the terms of the first base [row] by 3/2, those of the second by 2, those of the third by 5/2, and those of the fourth by 3, and so on to infinity.[40]

[40] "Nell'istessa maniera da quest'ultima Tavola riempita, si è fatta la Tavola pag. 18. moltiplicando il termine del vertice per 1, i termini della prima base per 3/2, quelli della seconda per 2, quelli della terza per 5/2, quelli della quarta per 3, e così gli altri in infinito." (Mengoli 1672).

1							
3/2	3/2						
2	$2a$	2					
5/2	15/4	15/4	5/2				
3	$12a/3$	6	$12a/3$	3			
7/2	105/16	35/4	35/4	105/16	7/2		
4	$96a/15$	12	$32a/3$	12	$96a/15$	4	
9/2	315/32	63/4	315/16	315/16	63/4	315/32	9/2

By introducing the inverse numbers again we arrive at the interpolated harmonic triangle, which is what he wished to calculate (see Fig. 11)

The numbers in this table correspond to the values of the table of interpolated figures. In modern notation, the areas or quadratures of these figures would be written as,

$$\int_0^1 \sqrt{x^n\,(1-x)^{(m-n)}}\mathrm{d}x = \frac{1}{(m/2+1)\binom{m/2}{n/2}} = B\left(\frac{n}{2}+1, \frac{m-n}{2}+1\right)$$

Here we have introduced the corresponding value of a Beta integral. In general, the quadrature of these figures between 0 and 1 would be a rational number for $m-n$ or n even integers, and when they are both odd integers one obtains multiples of the number π. For the second term of the second side (see Fig. 11)

$$\frac{1}{2a} = \frac{1}{(2)\binom{1}{1/2}} = \int_0^1 x^{1/2}(1-x)^{1/2}\mathrm{d}x = B(3/2, 3/2) = \frac{\pi}{8};$$

1							
2/3	2/3						
1/2	$1/2a$	1/2					
2/5	4/15	4/15	2/5				
1/3	$1/4a$	1/6	$1/4a$	1/3			
2/7	16/105	4/35	4/35	16/105	2/7		
1/4	$15/96a$	1/12	$3/32a$	1/12	$15/96a$	1/4	
2/9	32/315	4/63	16/315	16/315	4/63	32/315	2/9

Fig. 11 Interpolated harmonic triangle (Mengoli 1672)

(a)

$$1$$
$$2/3 \quad 2/3$$
$$1/2 \quad 1/2a \quad 1/2$$
$$2/5 \quad 4/15 \quad 4/15 \quad 2/5$$
$$1/3 \quad 1/4a \quad 1/6 \quad 1/4a \quad 1/3$$

(b)

$$FO.\ u.$$
$$FO.\ a^{1/2}. \quad FO.\ r^{1/2}.$$
$$FO.\ a. \quad FO.\ (ar)^{1/2}. \quad FO.\ r.$$
$$FO.\ a^{3/2}. \quad FO.\ (a^2r)^{1/2}. \quad FO.\ (ar^2)^{1/2}. \quad FO.\ r^{3/2}.$$
$$FO.\ a^2. \quad FO.\ (a^3r)^{1/2}. \quad FO\ .ar. \quad FO.\ (ar^3)^{1/2}. \quad FO.\ r^2.$$

(c)

$$\int_0^1 1\,dx$$
$$\int_0^1 x^{1/2}\,dx \quad \int_0^1 (1-x)^{1/2}\,dx$$
$$\int_0^1 x\,dx \quad \int_0^1 x^{1/2}(1-x)^{1/2}dx \quad \int_0^1 (1-x)dx$$
$$\int_0^1 x^{3/2}dx \quad \int_0^1 x(1-x)^{1/2}\,dx \quad \int_0^1 x^{1/2}(1-x)dx \quad \int_0^1 (1-x)^{3/2}\,dx$$
$$\int_0^1 x^2dx \quad \int_0^1 x^{3/2}(1-x)^{1/2}\,dx \quad \int_0^1 x(1-x)dx \quad \int_0^1 x^{1/2}(1-x)^{3/2}\,dx \quad \int_0^1 (1-x)^2\,dx$$

(d)

$$\mathrm{B}(1,1)$$
$$\mathrm{B}(3/2,1) \quad \mathrm{B}(1,3/2)$$
$$\mathrm{B}(2,1) \quad \mathrm{B}(3/2,3/2) \quad \mathrm{B}(1,2)$$
$$\mathrm{B}(5/2,1) \quad \mathrm{B}(2,3/2) \quad \mathrm{B}(3/2,2) \quad \mathrm{B}(1,5/2)$$
$$\mathrm{B}(3,1) \quad \mathrm{B}(5/2,3/2) \quad \mathrm{B}(2,2) \quad \mathrm{B}(3/2,5/2) \quad \mathrm{B}(1,3)$$

Fig. 12 **a** Interpolated harmonic triangle (Mengoli 1672), **b** interpolated *Tabula Formosa* (Mengoli 1672), **c** interpolated *Tabula Formosa* in modern notation, **d** table of Beta integrals

And for the fourth term of the second side (see Fig. 11)

$$\frac{1}{4a} = \int_0^1 x^{3/2}(1-x)^{1/2}\mathrm{d}x = B(5/2, 3/2) = B(3/2, 5/2) = \frac{\pi}{16}.$$

In these identities, it is worth noting that the value of "specious" number a is $4/\pi$, which was computed approximately by Mengoli (Massa-Esteve 1998).

In Fig. 12a–d, one may see the result for half-integers; that is, the table of quadratures or the interpolated harmonic triangle that has been obtained, the corresponding table of interpolated geometric figures in Mengolian notation and in modern notation, and the table of Beta integrals.

In addition to calculating the quadratures of the figures determined by $y = x^{n/2}(1-x)^{(m-n)/2}$, Mengoli goes a step further and states that the interpolation of the forms (geometric figures) and the value of their areas can be generalized for some other rational exponents.[41] Indeed, using the three properties of the combinatorial triangle that we have explained in the previous section, it is not difficult to obtain the quadratures of the figures determined by $y = x^{n/3}(1-x)^{(m-n)/3}$.

In this paragraph, we present an outline of the value of the quadrature of the circle, which is the term $FO.(ar)^{1/2}$ of the interpolated *Tabula Formosa*. Since $FO.(ar)^{1/2} = 1/2a$, the value of a, which is the ratio of the square to the quadrature of the circle,

[41] "61. And I have no doubt that if from the sequence of numbers 1, 2, 3, 4 and so forth, on doubling the terms and constructing the sequence 1, 1/2, 2, 5/2, 3, 7/2, 4 and so on, the Tables of numbers become doubled, and if on taking the geometric mean the Tables of forms [geometric figures] become (also) doubled, and the countless quadratures of the afore-mentioned three types of Theorems (see page…) are found: then from the same series of numbers, on tripling the terms and constructing the series 1, 4/3, 5/3, 2, 7/3, 8/3, 3, 10/3, 11/3, 4 and so on, the Tables of numbers can be triplicate; and on taking two geometric mean of the Rational [base] and the ordinates in the forms of the first Theorem, the Tables of forms [geometric figures] can be triplicate and other countless quadratures can be found; and so the Tables can be quadrupled and quintupled and more and more quadratures can be found." (Mengoli 1672).

must be calculated. Mengoli starts from the interpolated combinatorial triangle, in which appears for the first time the letter a. He was able to express the quadrature of the circle as an infinite product. Let us recall the interpolated combinatorial triangle we have just constructed.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & a & & 1 & & & & & \\
 & & & & 1 & & 3/2 & & 3/2 & & 1 & & & & \\
 & & & 1 & & 4a/3 & & 2 & & 4a/3 & & 1 & & & \\
 & & 1 & & 15/8 & & 5/2 & & 5/2 & & 15/8 & & 1 & & \\
 & 1 & & 24a/15 & & 3 & & 8a/3 & & 3 & & 24a/15 & & 1 & \\
1 & & 105/48 & & 7/2 & & 35/8 & & 35/8 & & 7/2 & & 105/48 & & 1
\end{array}
$$

The key point, to find the value of "a", is established as follows in Mengoli's words:

> 63. Let us return to table number 30 [the above table in this article], and consider there the sequences of terms on the second and penultimate sides; from which it is evident that the preceding terms are smaller than the succeeding ones; thus it seems convenient that each preceding term is smaller than the immediately succeeding ones; as it is likewise evident, that the first is smaller than the second, that is 1 is smaller than a.[42]

Mengoli expressed the relations between the terms in the table in this way. On one hand,

$a < 4a/3 < 24a/15 < \cdots$ On the other hand, $1 < 3/2 < 15/8 < \cdots$ and besides, $1 < a$.

Therefore, he takes the sequences of numbers on the second (and penultimate sides) and writes them in order

$1 < a < 3/2 < 4a/3 < 15/8 < 24a/15 < 105/48 < 192a/105 < \cdots$

From these infinitely many inequalities, Mengoli obtained a sequence of decreasing upper bounds of "a" as well as another sequence of increasing lower bounds of "a".

The first upper bound is $a < 3/2$.

From $4a/3 < 15/8$ he obtained the second upper bound $a < (3.3.5)/(2.4.4)$.

From $24a/15 < 105/48$ he obtained $a < (3.3.5.5.7)/(2.4.4.6.6) \ldots$ and so on.

Analogously, he obtained the lower bounds of "a".

From $3/2 < 4a/3$ he obtained $a > (3.3)/(2.4)$.

From $15/8 < 24a/15$ he obtained $a > (3.3.5.5)/(2.4.4.6) \ldots$ and so on.

[42] "63. Torniamo alla Tavola del num. 30, ed ivi consideriamo la serie de' termini del secondo, e penultimo lato; de quali è manifesto, che gli antecedenti sono minori de' susseguenti alterni: onde conviene ancora, che tutti per ordine gli antecedenti siano minori de' susseguenti; come è manifesto ancora, che il primo è minore del secondo, cioè 1, minore di a."(Mengoli 1672).

Mengoli summarized these two sequences of bounds inequalities with two corollaries, one for the upper bounds

> 67. From these two reflections, these two corollaries follow: First: that the ratio a to 1, the square [1] to the inscribed circle [$\pi/4$], is smaller than the product of any odd number by all the squares of the preceding odd numbers, (in relation) to the product of the first even number, which is 2, by all the squares of the other preceding even numbers [of the odd number].

For instance, taking the number 7, as an example of odd number, the antecedent of the ratio is 3.3.5.5.7, and the consequent of the ratio will be 2.4.4.6.6, so that the upper bound for "a" will be : $a <$ (3.3.5.5.7)/(2.4.4.6.6).

And another corollary for the lower bounds:

> 68. Second: that the ratio a to 1, the square [1] to the inscribed circle [$\pi/4$], is greater than the product of all the squares of the odd numbers, taken up to any even number, (in relation) to the product of two by that last even number, and by all the squares of the other preceding even numbers.[43]

Analogously, taking the number 8, as an example of even number, the antecedent of the ratio is 3.3.5.5.7.7, and the consequent of the ratio will be 2.4.4.6.6.8, so that the lower bound for "a" will be: (3.3.5.5.7.7)/(2.4.4.6.6.8) $< a$.

Therefore, the first delimitation proposed by Mengoli for $a = 4/\pi$ is

$$\frac{3.3.5.5.7.7.9.9\ldots}{2.4.4.6.6.8.8.10\ldots} < \frac{4}{\pi} = \frac{a}{1} = \frac{\text{Quadrat}}{\text{Cercle}} < \frac{3.3.5.5.7.7.9\ldots}{2.4.4.6.6.8.8\ldots}$$

By inverting all the fractions, he was able to delimit the quadrature of the circle ($\pi/4$). Finally, he obtained the approximation of the number π to 11 decimal places: 3.14159265246 (Massa-Esteve 1998).

It is worth pointing out that Wallis, in his *Arithmetica Infinitorum* (1655), started from the combinatorial triangle in the Proposition 132 (Wallis 1972), interpolating it to arrive at a table similar to Mengoli table in the Proposition 169 (Wallis 1972). The difference is that Wallis did not identify all the elements of the table directly with quadrature values for the corresponding figures. He spoke only of ratios that could be established between some values of the table, and from these ratios stated that the area of the circle could by found by interpolating. The geometric figure whose quadrature Wallis wished to find between 0 and 1, $y = (1 - x^2)^{1/2}$, is also different from Mengoli's geometric figure (Stedall 2001). We do not know whether Mengoli knew Wallis' work but the differences between the notation and the procedure are substantial. So, although it seems unlikely that Mengoli was familiar with Wallis' work, the fact that both worked in a similar way this quadrature is surprising.

[43] "67. Da queste due riflessioni seguono questi due Corollarij: Primo. Che a ad 1, il quadrato all inscrito circolo, è minore, che non è il prodotto da un numero dispare, per tutti li quadrati de' numeri precedenti dispari, in risguardo al prodotto dal primo pare, che è il binario, per tutti li quadrati de gli altri numeri precedenti pari. 68. Secondo. Che a ad 1, il quadrato all'inscritto circolo, è maggiore, che non è il prodotto da tutti li quadrati de' numeri dispari, presi fino ad un qualche pare, in risguardo al prodotto da quell'ultimo pare, e dal binario, e da tutti li quadrati de gli altri numeri pari precedenti." (Mengoli 1672).

Nevertheless, in the calculation of Beta in the earlier work (see the introduction), Euler's aim was to find the general term for a series consisting of the terms 1, 2, 6, 24…that is, the general term for the sequence "$n!$" by means of an algebraic expression. He realized that this was impossible after he noticed that in the case $n = 1/2$ he obtained an infinite product that reminded him of the infinite product in Wallis' work. He then used these integrals because the terms of these sequence contains terms that depend on quadratures. In fact these integral formulas were essential for Euler to express the general term of this sequence and calculate the value of $(1/2)! = \sqrt{\frac{\pi}{4}}$ (see Ferraro 1998; Delshams and Massa 2008).

3 Mengoli's properties and theorems for quadratures

In this section we describe two kinds of achievements, properties and theorems, obtained by Mengoli from his triangular tables.

3.1 Property of sums

The first property is the property of the infinite sums of the harmonic triangle, which is the one of the most original. It can be stated as: "The infinite sum of Beta integrals on one side of the harmonic triangle gives the first Beta function of the previous side". In modern notation, this would be

$$\sum_{n=0}^{\infty} B(m+1, n+1) = \sum_{n=0}^{\infty} \frac{1}{(m+n+1)\begin{pmatrix} m+n \\ n \end{pmatrix}} = \frac{1}{m} = B(m, 1).$$

Mengoli proved this property in the harmonic triangle he constructed in the *Geometriae*. For the proof of this property, Mengoli used the value of the infinite sums of series he had calculated and proved in an earlier work, *Novae Quadraturae Arithmeticae seu de Additione Fractionum*(1650) (referred to henceforth as *Novae*).[44] Let us recall the harmonic triangle,

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1/2 & & 1/2 & & & & & \\
 & & & & 1/3 & & 1/6 & & 1/3 & & & & \\
 & & & 1/4 & & 1/12 & & 1/12 & & 1/4 & & & \\
 & & 1/5 & & 1/20 & & 1/30 & & 1/20 & & 1/5 & & \\
 & 1/6 & & 1/30 & & 1/60 & & 1/60 & & 1/30 & & 1/6 & \\
1/7 & & 1/42 & & 1/105 & & 1/140 & & 1/105 & & 1/42 & & 1/7
\end{array}$$

[44] In this work Mengoli worked with infinite series, adding them together and giving them properties. Explanations and proofs for these calculations can be found in Giusti (1991)and Massa-Esteve (1998)

The property is obtained by adding the terms of the sides in an infinite way, so that the side whose first term is $1/(m+1)$, has as its sum $1/m$ with $m \geq 1$. For example:

$1/2 + 1/6 + 1/12 + 1/20 + \cdots = 1;$
$1/3 + 1/12 + 1/30 + 1/60 + \cdots = 1/2;$
$1/4 + 1/20 + 1/60 + 1/140 + \cdots = 1/3;$

In modern notation, the first sum in integral terms would be:

$$\int_0^1 x\mathrm{d}x + \int_0^1 x(1-x)\,\mathrm{d}x + \int_0^1 x(1-x)^2\mathrm{d}x + \int_0^1 x(1-x)^3\mathrm{d}x + \cdots = 1/2 + 1/6 + 1/12 + 1/20 + \cdots = 1$$

Mengoli explains that he arrived at the value of this infinite sum of quadratures on the second side from the results obtained by the indivisibles and from Proposition 17 (Mengoli 1650) in his *Novae*. In Proposition 17 in the *Novae*, he had proved that $\sum_{n=0}^{\infty} \frac{1}{(n+1)\,(n+2)} = 1/2 + 1/6 + 1/12 + \cdots = 1$.

Subsequently in the *Geometriae*, after calculating the value of the quadratures by the method of the indivisibles, Mengoli placed them in a triangular table and added up the values of the sides in order to obtain a new quadrature. With this second side he obtained:

$$\sum_{n=0}^{\infty} \int_0^1 x(1-x)^n \mathrm{d}x = \sum_{n=0}^{\infty} \frac{1}{(n+2)\binom{n+1}{1}} = 1 = \int_0^1 1\mathrm{d}x.$$

He also added up the terms on the third side of the harmonic triangle

$$\int_0^1 x^2\mathrm{d}x + \int_0^1 x^2(1-x)\,\mathrm{d}x + \int_0^1 x^2(1-x)^2\mathrm{d}x + \cdots = 1/3 + 1/12 + 1/30 + \cdots = 1/2.$$

In the Proposition 8 of book 2 of the *Novae*, Mengoli calculated sum of the following infinite series (Mengoli 1650)

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)(n+3)} = 1/6 + 1/24 + 1/60 + \cdots = \frac{1}{4};$$

If all the terms are multiplied by 2, we obtain $1/3, 1/12, 1/30, \ldots$ which added infinitely yield 2/4; that is, 1/2. Expressed in combinatorial numbers, this is

$$\sum_{n=0}^{\infty}\int_0^1 x^2(1-x)^n \mathrm{d}x = \sum_{n=0}^{\infty}\frac{1}{(n+3)\binom{n+2}{2}} = \frac{1}{2} = \int_0^1 x\mathrm{d}x.$$

Expressed not in letters but only verbally, Mengoli generalized these sums of series thus:

> And in general, finding the figure in which the ordinates are all the powers of the abscissae, and successively all the figures in which the ordinates are the product of the same powers of the abscissae and all the possible powers of the residues, all added together, is equal to the figure in which the ordinates are all the powers of the abscissae of the closest lower order.[45]

In modern notation and generalizing, the property of sums would be:[46]

$$\sum_{n=0}^{\infty}\int_0^1 x^m(1-x)^n \mathrm{d}x = \frac{1}{(m+1)\binom{m}{0}} + \frac{1}{(m+2)\binom{m+1}{1}} + \frac{1}{(m+3)\binom{m+2}{2}} + \cdots = \frac{1}{m} = \int_0^1 x^{m-1}\mathrm{d}x.$$

3.2 "Bella proprietà"

We now point out an important property that Mengoli called a "Bella proprietà" [Beautiful property] in the *Circolo* and which he explained after constructing the interpolated harmonic triangle,

> And, it is a noteworthy beautiful property of Table of page 19, that each term is as a abdomen, where laterally two legs are hanging, with two terms, one the knee and the other the foot, and the abdomen is the sum of its two feet [47]

In modern notation and in terms of Beta integral it would be written as: $B(p,q) = B(p+1,q) + B(p,q+1) \quad p,q$ half-integer and positive numbers.[48] One example of the "Bella Proprietà", in the interpolated harmonic triangle (see Figs. 11, 12d) is: $B(3/2,3/2) = B(5/2,3/2) + B(3/2,5/2)$; $\frac{1}{2a} = \frac{1}{4a} + \frac{1}{4a}$.

[45] "Et generaliter inveni, figuram, in qua ordinatate sunt omnes potestates abscissarum, & deinceps omnes figuras, in quibus ordinatae sunt productae sub ijsdem potestatibus abscissarum, & sub residuarum potestatibus omnifariam, simul aggregatas, aequales esse figurae, in qua ordinatae, sunt omnes potestates abscissarum ordinis proximè inferioris." (Mengoli 1659).

[46] Generalizing for any term of the side $\sum_{q=l}^{\infty} B(p,q) = B(p-1,l)$, $p>1, l>0$.

[47] "39. Ed è notabile una bella proprietà della Tavola pag. 19. che ciascun termine è come un ventre, onde pendono lateralmente due gambe, con due termini l'una, ginocchio, e piede; ed è il ventre la somma de' due suoi piedi." (Mengoli 1672)

[48] Indeed this formula is valid for any p, q real and positive.

3.3 Two other properties

The last two properties that we describe were called by him *convenienze* (Mengoli 1672)[49] and they are valid for the rows of both the combinatorial and the interpolated combinatorial triangle. In modern notation and in terms of combinatorial numbers, these two properties would appear as;

$$(1)\ \binom{m}{n} = \binom{m-1}{n-1} + \binom{m-1}{n}; \quad (2)\ \frac{\binom{m}{n}}{\binom{m}{n+1}} = \frac{1+n}{m-n}$$

for real m and half-integer n.

3.4 Theorems for quadratures

We now describe the three kinds of theorems stated and proved by Mengoli after constructing the interpolated harmonic triangle in the *Circolo*. Mengoli explained that the co-ordination of the two interpolated triangles, that of the geometric figures (interpolated *Tabula Formosa*, see Fig. 8) and that of the values of their quadratures (interpolated harmonic triangle, see Fig. 11), is a theorem containing infinitely many theorems about quadratures, since the number of terms of the triangular tables are countless and can grow indefinitely.

> That is a Theorem, which contains innumerable theorems all of them on quadratures, in the same way as the terms of triangular tables are innumerable, since they [terms] can grow indefinitely.[50]

---

[49] The first corresponds to the standard formation of the combinatorial triangle. Mengoli explains that each term is the sum of the two terms in the spaced out above row.

$$\begin{array}{ccccccccccc} & & 1 & & 3/2 & & 3/2 & & 1 & & \\ & 1 & & 4a/3 & & 2 & & 4a/3 & & 1 & \\ 1 & & 15/8 & & 5/2 & & 5/2 & & 15/8 & & 1 \end{array}$$

Thus $5/2 = 1 + 3/2$, $8a/3 = 4a/3 + 4a/3$.
The second property enunciated by Mengoli concerns also the terms of the rows in the table. He explains that, just as in the table of combinatorial numbers, it is verified in each row that the ratio of the first term to the second gives the ratio of the other terms by adding one unity to the predecessor and subtracting one unity from the successor, likewise in each row of the interpolated table, from the first term to the third the ratios are obtained by adding 1/2 to the predecessor of the first ratio and subtracting 1/2 from the successor. Consider for example the eighth row:
$\frac{1+(1/2)}{4-(1/2)} = \frac{3/2}{7/2} = \frac{192a/105}{64a/15}$; $\frac{(3/2)+(1/2)}{(7/2)-(1/2)} = \frac{2}{3} = \frac{4}{6}$; $\frac{2+(1/2)}{3-(1/2)} = \frac{5/2}{5/2} = \frac{64a/15}{64a/15}$.

[50] "Questo è un Teorema, che contiene innumerabili Teoremi tutti di Quadrature, come sono innumerabili i termini delle Tavole Triangolari, che possono crescere in infinito." (Mengoli 1672).

He classified the values in the interpolated harmonic triangle into three kinds of theorems for quadratures $\int_0^1 \sqrt{x^n (1-x)^{(m-n)}}\mathrm{d}x$ depending on the exponents inside the integrals.

The first kind of theorems for the interpolated harmonic triangle refers to the quadratures proved in the sixth book of the *Geometriae*; which give rise to theorems in which there is no square root in the integrals since the exponent n and $m-n$ are even. These are the figures *FO. a*, *FO. r*, $FO.a^2$, etc, in the interpolated *Tabula Formosa*, determined by the curves $y = x$, $y = 1 - x$, $y = x^2$, etc., the quadratures of which take the values 1/2, 1/2, 1/3, etc. in the interpolated harmonic triangle (see the Sect. 2.2), since they are homologous terms.

The second kind of theorems in the interpolated harmonic triangle refers to those quadratures where the exponent n is an even number or the exponent $m - n$ is even. Mengoli first explained that the ratio between the terms of the interpolated *Tabula Formosa* is equal to the ratio between the homologous terms of the interpolated harmonic triangle. So, the quadrature of the figure $FO\,.\,a^{3/2}$, determined by the curve $y = x^{3/2}$ is homologous to the value 2/5, that of $FO\,.\,a^{1/2}$, determined by the curve $y = x^{1/2}$ is homologous to the value 2/3, and that of $FO.\ (a^2r)^{1/2}$, determined by the curve $y = (x^2(1-x))^{1/2}$ is homologous to the value 4/15. Moreover, Mengoli deduced that some of these results are true in specific examples, by using the method of inscribed and circumscribed figures.

The third kind of theorems in the interpolated harmonic triangle refers to those quadratures where the exponents n and $m-n$ are odd, and in which the specious number "a" (Mengoli 1672) appears. First, Mengoli stated the proportion between the homologous terms again. Then he deduced that the quadrature of the figure $FO.\ (a^3r)^{1/2}$, determined by the curve $y = (x^3(1-x))^{1/2}$ is homologous to the value $1/4a = \pi/16$ (in modern notation, B (5/2,3/2)) and that the quadrature of the semi-circle, which corresponds to the figure $FO.(ar)^{1/2}$, determined by the curve $y = (x(1-x))^{1/2}$, is homologue to the value $1/2a = \pi/8$ (in modern notation B (3/2, 3/2)) (Mengoli 1672).

Although Mengoli's justification of the quadratures of the second and the third kinds was made with the help of inscribed and circumscribed figures, he claimed later on that these proofs are not really necessary

> 58. But these proofs (of the inscribed and circumscribed figures) of the second and third type Theorems are not explained in order to do science, but only to provide an example of what I state, and to express better my idea [*concetto*]: like when the Theorems of Euclid's second Element are explained with numbers, this is not done to prove them, or make science of the conclusions, but rather to explain the meaning of the words.[51]

According to Mengoli, in order to prove the values of the quadratures for any figure of each type, it suffices to prove those of the first kind already analyzed, and to under-

[51] "58. Ma di queste dimostrationi de' Teoremi della seconda e terza classe, non sò alcun conto per fare scienza, ma solo per dare essempio di chel ch'io dico, e per dichiarare meglio il mio concetto; come quando si spegiano i Teoremi del secondo Elemento d'Euclide per numeri, ciò non si fà per dimostrare, e fare scienza delle conclusioni, ma per ispianare il senso delle parole." (Mengoli 1672).

stand the triangular tables and their co-ordination since the first table is reproduced (*replicata*) in the last table and the construction preserves the proportion between the terms displayed in the same place. He explained these ideas in this way:

> 59. Notwithstanding, for the intellect to be convinced of the truth of the quadratures proposed, no other thing is required than the proofs of the Theorems of the first type (proof by natural numbers), with the comprehension of the Triangular Tables, which we have thus far conceived, and suitably co-ordinated in order to understand all the many quadratures of the same first type (construction of the table of values): it being sufficient for comprehending the truth, to understand that the quadratures of the other two Types are also found in the same co-ordination.[52]

In fact we can deduce that two triangular tables are coordinated if its terms are homologous. Consequently the terms of the second and third kinds of triangular tables, geometric figures and the values of their quadratures, are also homologous terms, that is to say, preserves the same proportion. So, we can claim that the numerical computation of interpolated harmonic triangle allowed Mengoli to show values of countless quadratures of geometric figures depending on the corresponding exponents.

4 Conclusions

Mengoli's aim was to find an algorithm that would enable him to calculate many quadratures at the same time, including some of them, such as the quadrature of the circle, that had never been calculated before. He first attempted this in the *Geometriae* with the method of indivisibles, but he was only able to obtain the quadratures of geometric figures determined by algebraic expressions of the form $y = x^n(1-x)^{m-n}$ with exponents m and n that are natural numbers. He displayed these geometric figures in a triangular table (*Tabula Formosa*). He later identified these geometric figures explicitly with the values of their areas, which were also displayed in another triangular table (now called the harmonic triangle) by means of a proof that employed the theory of quasi proportions and, to some extent, the method of exhaustion. In the *Circolo*, for half-integer values of the exponents, he interpolated both tables, and was thus able to calculate the area of the figure describing a semi-circle as well as many other areas not computed before. It is worth remembering that, prior to Newton and Leibniz, when mathematicians wished to obtain quadratures of geometric figures determined by algebraic expressions with roots, they had to do it case by case.

This work stresses the importance given by Mengoli to the combinatorial triangle as a computational tool. One may ask how Mengoli could say that the properties of the combinatorial triangle can remain valid in the interpolated combinatorial triangle. Even more, one may ask how the result of the quadratures proved by Mengoli with

52 "59. Anzi perche l'intelletto resti determinato nella verità delle quadrature proposte, non bisognano altre, che le dimostrationi de' Teoremi della prima classe, con l'intelligenza delle Tavole Triangolari, che habbiamo concepite fin quì, e convenientemente coordinate, à comprendere tutte le innumerabili quadrature della medesima prima Classe: bastando à intendere il vero, l'intendere, che nella stessa coodinatione, convengono del pari le quadrature ancora dell altre due Classi." (Mengoli 1672).

natural exponents (the harmonic triangle) will also remain valid for rational exponents (the interpolated harmonic triangle).

The relation between the *Tabula Formosa* and the harmonic triangle (see the Sect. 1.2) preserves the proportions between the terms situated in the same place in each of the tables, terms referred to by Mengoli as "homologous terms". Thus, the square of the vertex is homologous to unity, and the figures in the first row are homologous to 1/2, and so on. Likewise with rational exponents, Mengoli makes clear that this relation, referred to by him as "the co-ordination of the two tables", is maintained since the interpolated tables (geometric figures and values of areas) are reproduced inside the first tables. Moreover, the proportion between the geometric figures (interpolated *Tabula Formosa*) and the homologous values of their areas (interpolated harmonic triangle) is preserved thanks to their construction (see the Sect. 3.4). Through these new homologous terms, Mengoli extended the former dependence found between the values of integrals and their exponents to rational exponents.

While it is true that the complicated notation and the intricate calculations involved in Mengoli's method hampered it from reaching a wider audience (see Footnote 5), his method is nevertheless very fertile, since it makes it possible to find both known results and many new ones. By employing the interpolated combinatorial triangle, the dependence between the integrals and their exponents appears naturally. Furthermore, Mengoli's definition of the triangular tables, the symmetry of them, and the regularity of its rows enabled him to deduce directly the main properties of what is known today as the Beta function. Thus Mengoli's statements, for the natural numbers and half-integers p and q, in modern notation, can be expressed as [see the identity (2)]:

$$B(p, q) = \frac{1}{(p+q-1)\binom{p+q-2}{p-1}};$$

In particular, for the quadrature of the circle [see the identity (3)]:

$$B(3/2, 3/2) = \frac{1}{(3/2+3/2-1)\binom{3/2+3/2-2}{3/2-1}} = \frac{1}{(2)\binom{1}{1/2}} = \frac{1}{2a}.$$

Therefore, when the history of the Beta integral is analyzed, it will be necessary to quote Mengoli as one of its pioneers, since he hit upon the algorithmic power of the relations between these integral values. Thus, assigning to each quadrature the corresponding value of the interpolated harmonic triangle, we are able to state that, though it was only for natural numbers and half-integers, Mengoli had a clear conception (definition and properties) of these kinds of values of quadratures, now called values of the Beta function.

Acknowledgments We are grateful to Giovanni Ferraro who provided us with many valuable references to Euler's Beta function. We are very grateful to Jacqueline Stedall, Eberhard Knobloch and again to Giovanni Ferraro for their valuable comments on a previous version of this article. Also, we are grateful to Dolors Magret who made some remarks concerning language.

References

Andersen, Kirsti, 1984/1985. Cavalieri's method of indivisibles. *Archive for History of Exact Sciences* 31:291–367.

Auger, Léon, 1962. *Un savant méconnu: Gilles Personne de Roberval (1602–1675)*. Paris: Libraire A. Blanchard.

Baroncini, Gabrielle, and Cavazza, Marta. 1986. *La corrispondenza di Pietro Mengoli*. Florence: Olschki.

Bosmans, Henry. 1924. Sur l'oeuvre mathématique de Blaise Pascal. *Mathesis* 38:1–59.

Cajori, Florian. 1928–1929. *A history of mathematical notations*, Vol. 2. Chicago: Open Court (Reprinted by Dover, 1993.)

Calinger, Ronald. 1996. Leonhard Euler: the first St. Petersburg years (1727–1741). *Historia Mathematica* 23:121–166.

Cifoletti, Giovanna. 1990. La méthode de Fermat: son statut et sa difussion. In: *Cahiers d'histoire et de philosophie des sciences*, nouvelle série, Vol. 33. Paris: Société française d'histoire des sciences et des techniques.

Delshams, Amadeu, and Massa, Ma Rosa. 2008. Consideracions al voltant de la funció Beta a l'obra de Leonhard Euler (1707–1783). *Quaderns d'Història de l'Enginyeria* IX:59–82.

Descartes, René. 1954. *The geometry of René Descartes,* Eds. D.E. Smith and M.L. Latham. New York: Dover.

Edwards, Anthony William F. 2002. (1^a ed. 1987). *Pascal's Arithmetical triangle. The story of a mathematical idea*. New York: Oxford University Press.

Euler, Leonhard. 1730–1731. De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt. In *Euler Opera Omnia* 14(I):1–24.

Euler, Leonhard. 1739. De productis ex infinitis factoribus. In: *Euler Opera Omnia*, 14(I):260–290.

Euler, Leonhard. 1753. De expressione integralium per factores. In: *Euler Opera Omnia*, 17(I):233–267.

Fermat, Pierre de. 1891–1922. *Oeuvres*. Eds. P. Tannery and H. Charles, 4 vols. 7 spp. Paris: Gauthier Villars.

Ferraro, Giovanni. 1998. Aspects of Euler's theory of series. *Historia Mathematica* 25:290–317.

Ferraro, Giovanni. 2000. Functions, functional relations and the laws of continuity in Euler. *Historia Mathematica* 27:107–132.

Ferraro, Giovanni. 2004. Differentials and differential coefficients in the Eulerian foundations of the calculus. *Historia Mathematica* 31:34–61.

Ferraro, Giovanni. 2008. The integral as anti-differencial. An aspect of Euler's attempt to transform the calculus into an algebraic calculus. *Quaderns d'Història de l'Enginyeria* IX:25–58.

Fraser, Craig. 1989. The calculus as algebraic analysis: Some observations on mathematical analysis in the 18th century. *Archive for History of Exact Sciences* 39:317–335.

Giusti, Enrico. 1980. *Bonaventura Cavalieri and the theory of indivisibles*. Bologna: Edizioni Cremonese.

Giusti, Enrico. 1991. Le prime ricerche di Pietro Mengoli. La somma delle serie. In *Geometry and Complex Variables: Proceedings of an International Meeting on the Occasion on the IX Centennial of the University of Bologna*, ed. S. Coen, 195–213. New York: Dekker.

Gray, Jeremy. 1985. Leonhard Euler: 1707–1783. *Janus: Revue internationale de l'histoire des sciences, de la médecine, de la pharmacie et de la technique* LXXII, 1–3:171–192.

Gregory, James. 1939. *Tercentenary memorial volume*, ed. H. W. Turnbull, London: Bell.

Hérigone, Pierre. 1642–1644. *Cursus mathematicus, Nova, Brevi et clara methodo demonstratus, Per NOTAS reales & universales, citra usum cuiuscunque idiomatis, intellectu faciles*. (Second editions), 6 vols. Paris: Simeon Piget.

Hobson, Ernest William. 1913. *Squaring the circle. A history of the problem*. Cambridge: Cambridge University.

Jami, Catherine. 1988. Une histoire chinoise du nombre pi. *Archive for History of Exact Sciences* 38, 1:39–50.

Leibniz, Goffried, W. 1672–1676. *Sämtliche Schriften und Briefe*, series VII: *Mathematische Schriften*, Vol. III: *Differenzen-Folgen-Reihen*, Berlin, 2003, 735–748.

Mahoney, Michael Sean. 1973. *The mathematical career of Pierre de Fermat*. Princeton: Princeton University (reprinted 1994).

Malet, Antoni. 1996. *From indivisibles to infinitesimals, Studies on seventeenth century mathematizations of infinitely small quantities.* Barcelona: Universitat Autònoma de Barcelona.

Massa-Esteve, Ma Rosa 1994. El mètode dels indivisibles de Bonaventura Cavalieri. *Butlletí de la Societat Catalana de Matemàtiques* 9:68–100.

Massa-Esteve, Ma Rosa. 1997. Mengoli on "Quasi Proportions".*Historia Mathematica* 24, 2:257–280.

Massa-Esteve, Ma Rosa. 1998. *Estudis matemàtics de Pietro Mengoli (1625–1686): Taules triangulars i quasi proporcions com a desenvolupament de l'àlgebra de Viète.*Tesi Doctoral. Barcelona: Universitat Autònoma de Barcelona. http://www.tdx.cat/TDX-0506108-144848.

Massa-Esteve, Ma Rosa. 2003. La théorie euclidienne des proportions dans les "Geometriae Speciosae Elementa" (1659) de Pietro Mengoli. *Revue d'Histoire des Sciences* 56, 2:457–474.

Massa-Esteve, Ma Rosa. 2006a. Algebra and geometry in Pietro Mengoli (1625–1686). *Historia Mathematica* 33:82–112.

Massa-Esteve, Ma Rosa. 2006b. *L'Algebrització de les Matemàtiques. Pietro Mengoli* (1625–1686). Barcelona: Institut d'Estudis Catalans.

Massa-Esteve, Ma Rosa. 2006c. La algebrización de las Matemáticas. L'algèbre de Pierre Hérigone (1580–1643). In *Actas del IX Congreso de la Sociedad Española de Historia de las Ciencias y de las Técnicas*, Tomo 1, Cádiz, 183–197.

Massa-Esteve, Ma Rosa. 2007. Leonhard Euler (1707–1783): L'home, el creador i el Mestre. *Mètode*, 35–38. Valencia: Universitat de València.

Massa-Esteve, Ma Rosa. 2008. Symbolic language in early modern mathematics: the algebra of Pierre Hérigone (1580–1643), *Historia Mathematica* 35:285–301.

Mengoli, Pietro. 1650. *Novae Quadraturae Arithmeticae seu de Additione Fractionum*, Bologna.

Mengoli, Pietro. 1659. *Geometriae Speciosae Elementa.* Bologna.

Mengoli, Pietro. 1672. *Circolo*. Bologna.

Natucci, A. 1970–1991. Mengoli. In *Dictionary of scientific biography*, ed. C.C. Gillispie, Vol. 9, 303–304. New York: Scribner's.

Pascal, Blaise. 1954. *Oeuvres complètes*. Paris: Gallimard.

Probst, Siegmund. 2009. The Reception of Pietro Mengoli's Work on Series by Leibniz (1672–1676). In Proceedings of Joint International Meeting UMI-DMV, Perugia, 18–22 June 2007.

Seidenberg, Abraham. 1981. The ritual origin of the Circle and Square. *Archives for History of Exact Sciences* 25:269–327.

Stedall, Jacqueline, 2001. The discovery of wonders: reading between the lines of John Wallis's arithmetica infinitorum. *Archive for History of Exact Sciences* 56, 1–28.

Volkov, Alexei, 1997. Zhao Youqin and his calculation of pi. *Historia Mathematica* 24:301–331.

Walker, Evelyn. 1986. *A study of the Traité des Indivisibles of... Roberval.* New York: Columbia University.

Wallis, John. 1972. *Opera mathematica*, 3 vols. *Arithmetica infinitorum*, Vol. 1. New York: Olms.

Youschkevitch, Adolf P. 1970–1991. Euler. In *Dictionary of scientific biography*, ed. C.C. Gillispie Vol. 9, 467–484. New York: Scribner's.