Commun. Math. Phys. (2025) 406:146
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-025-05342-1

Communications in Mathematical Physics

Generic Global Diffusion for Analytic a Priori Unstable Systems

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Received: 16 December 2024 / Accepted: 2 May 2025

Abstract: We show that given a general a priori unstable Hamiltonian

$$\frac{1}{2}p^2 + V(q) + G(I) + \epsilon h(p, q, I, \varphi, t),$$

where h is a generic Mañé analytic function and ϵ is small enough, there is an orbit for which the momentum I changes by any arbitrarily prescribed value. We call this phenomenon as global diffusion since the size of the change in I is independent of both ϵ and h. The fact that the pendulum and rotor variables are uncoupled is used essentially in our proof. The proof is based on simple and constructive geometrical methods, carefully studying the reduced Poincaré functions of the problem which generate the corresponding scattering maps.

1 Introduction

For a nearly integrable Hamiltonian

$$H_\epsilon(I, \varphi) = H_0(I) + \epsilon H_1(I, \varphi), \tag{1}$$

a large part of the phase space is occupied by invariant KAM tori, making the system remarkably stable. Arnold diffusion is the study of topological instability in the complement of these KAM tori. The instability is created by resonances of the unperturbed frequency $\partial_I H_0(I)$. Indeed, in the first example of Arnold diffusion, Arnold [1] considers the system

$$H_\epsilon = \frac{1}{2}(I_1^2 + I_2^2) + \epsilon(\cos\varphi_1 - 1) + \epsilon\mu(\cos\varphi_1 - 1) f(\varphi_2, t), \tag{2}$$

and proves that for suitable f and $0 < \mu \ll \epsilon$, there exists an orbit whose I_2 component change for an arbitrary large distance along the resonance $\{I_1 = 0\}$. Since μ is taken

Published online: 30 May 2025

much smaller than ϵ, after a rescaling we may assume $\epsilon = 1$ in (2). This type of systems are called *a priori unstable* because the hyperbolic structure that leads to instability exists even at $\mu = 0$. The original problem (1) is called *a priori stable*, since the unperturbed system is completely integrable.

We have seen much progress in proving Arnold diffusion holds for *generic C^r a priori stable* systems *for a convex H_0*, while the non-convex case remains wide open. We refer to [5,6,15,27,29,30] for a non-exhaustive list of references. Those results are built on decades of progress on *a priori unstable* systems, see [2,3,7,8,12,13,25,31,32] and reference therein.

For real analytic systems, however, generic Arnold diffusion for *a priori stable* systems remain wide open. A large body of literature is dedicated to analytic examples of diffusion (for example [26,33]) and concrete systems like the N-body problems (for example [9,16]). A related problem is the instability around elliptic fixed points or KAM tori, see [19–21]. For generic analytic perturbation of *a priori chaotic* systems, we mention the works of Gelfreich-Turaev [22] and Clarke-Turaev [10]. For a priori unstable systems, Delshams-Schaefer [17,18] studied the diffusion for a finite parameter family of perturbations.

The most relevant work to this paper is [4] by Chen-de la Llave, who proved that for an *a priori unstable* system of the type

$$
\begin{aligned}
&H_\epsilon = H_0(p, q, I) + \epsilon h(p, q, I, \varphi, t), \quad H_0(p, q, I) = G(I) + \sum_{i=1}^{d} \pm \left(\frac{1}{2}p_i^2 + V_i(q_i)\right), \\
&\qquad p \in \mathbb{R}^n,\ q \in \mathbb{T}^n,\ I \in \mathbb{R}^d, \varphi \in \mathbb{T}^d, t \in \mathbb{T},
\end{aligned} \tag{3}
$$

for a Mañé generic analytic perturbation H_1 (precise definition to be given later), there exists a diffusion orbit $(p, q, I, \varphi)(t)$ and $T > 0$ such that

$$|I(T) - I(0)| > \rho(H_1).$$

The point is that the magnitude of the topological instability does not vanish as $\epsilon \to 0$.

The main result of our paper is that in a system of type (3), for $n = 1$ and $d = 1$, one can achieve *large-scale* diffusion for a Mañé generic analytic perturbation independent of H_1. Indeed, our main result claims that given *any* $I^- < I^+$, for a Mañé generic analytic potential H_1, *for all* $\epsilon < \epsilon_0(H_1)$, the system $H_0 + \epsilon H_1$ admits an orbit $(p, q, I, \varphi)(t)$ and $T > 0$ such that

$$I(0) < I^- < I^+ < I(T).$$

We call this large drift of the variable I *global diffusion*, compared to the local-scale diffusion of [4]. In particular, it follows from our result that Arnold's Theorem in [1] remains true for a generic analytic perturbation of the type $h(\varphi_1, \varphi_2, t)$ instead of the special perturbation $(\cos\varphi_1 - 1) f(\varphi_2, t)$ chosen by Arnold.

One of the goals of this paper is to show generic global instability in a simple way so that it is accessible to a wider audience. To avoid technical complications, we have restricted ourselves to the one-dimensional case both with regard to the pendulum variables p, q, and the action-angle variables I, φ.

On the one hand, following the philosophy that more hyperbolic variables generate more instability paths, we believe that, by appropriately modifying the Melnikov Potential used, the method followed in this work could be applied to the case in which this hyperbolicity is produced by several pendulums, although this would affect the genericity of the unperturbed model.

On the other hand, the introduction of multi-dimensional action variables entails the treatment of several possible instability directions as well as the appearance of resonances, and here the method presented could not be applied directly.

Besides the a-priori unstable system (3), it is interesting to consider the *coupled* a priori unstable systems which include also a term coupling I, p. These coupled systems seems more relevant in the study of normal forms in near integrable systems, see, for instance [14]. In these coupled a priori unstable systems, the scattering map involves a phase-shift. We will not consider this in this paper. Moreover, since the name a priori unstable has already been used many times in the literature, we have preferred not to include the adjective "uncoupled" throughout this paper for the a priori unstable systems we are considering.

Our proof uses the scattering map approach (see [12,13,23,24]). The diffusion orbit travels close to a normally hyperbolic invariant manifold (NHIM), and the dynamics can be described by the scattering map (homoclinic excursion) and inner map (dynamics restricted to the NHIM). The scattering map can be computed to the first order by means of the Melnikov potential $\mathcal{L}_h(I, \varphi, s)$ introduced in (6). Our approach is twofold:

- Generically, for every fixed I the Melnikov potential $\mathcal{L}_h(I, \cdot, \cdot)$ admits a non-degenerate local minimum. These local minima are obtained from local extension of the global minima.
- Near these local minima of the Melnikov potential, we show that there exists a pseudo-orbit of the scattering map and the restricted dynamics to the invariant manifold whose I component is increasing. We call this the "Ascending Ladder". We then show that there exists a trajectory of the Hamiltonian system which shadows the ascending ladder.

It is important to emphasize that these previous conditions that guarantee genericity are fully verifiable in specific systems from Physics, Engineering or other applied fields, and allow the explicit calculation of the pseudoorbits of the scattering map which are then shadowed by trajectories of the system.

In the approach followed, no quantitative estimates of diffusion time are made nor is there an attempt to optimize them, since our goal is to propose a very general, simple, and constructive diffusion mechanism. In pursuit of simplicity for the reader, this paper is almost self-contained, except for the theory of scattering maps and the geometric mechanism of diffusion (Proposition 3.1, equation (8), and Lemma A.1), for which specific references are given.

2 Formulation of the Main Result

Consider the Hamiltonian

$$H_\epsilon(p, q, I, \varphi, t) = \frac{1}{2}p^2 + V(q) + G(I) + \epsilon h(p, q, I, \varphi, t), \tag{4}$$

where

$$(p, q, I, \varphi, t) \in M := \mathbb{R} \times \mathbb{T} \times \mathbb{R} \times \mathbb{T} \times \mathbb{T}.$$

Assume that

(1) V has a non-degenerate global maximum; without loss of generality, suppose it is at $q = 0$.
(2) For the Hamiltonian system $\frac{1}{2}p^2 + V(q)$, the stable and unstable manifolds of $(0, 0)$ intersect along a homoclinic trajectory

$$\{(p_0(t), q_0(t)) : t \in \mathbb{R}\}. \tag{5}$$

(3) The frequency map $I \mapsto \omega(I) := G'(I)$ admits the following property: For every interval $(I_1, I_2) \subset [I^-, I^+]$, $\omega((I_1, I_2))$ contains a nontrivial interval. Note that in particular, this is satisfied if $\omega(I)$ never vanishes.

Definition 1. Let $U \subset M$ be an open set. We define its σ-complex neighborhood to be

$$\mathcal{B}_\sigma(U) = \{\tilde{z} = (p, q, I, \varphi, t) \in \mathbb{C} \times (\mathbb{C}/\mathbb{Z}) \times \mathbb{C} \times (\mathbb{C}/\mathbb{Z}) \times (\mathbb{C}/\mathbb{Z}) : \operatorname{dist}(\tilde{z}, U) < \sigma\}.$$

Let $\mathcal{A}_\sigma(U)$ denote the space of all real analytic functions on U that can be extended to a bounded analytic function on $\mathcal{B}_\sigma(U)$. Equipped with the supremum norm, this is a Banach space. Let

$$Q = \mathbb{T} \times \mathbb{T} \times \mathbb{T}$$

denote the configuration space, i.e., the space of the variables (q, φ, t). We will also consider the complex strip

$$\mathcal{D}_\sigma = \{(q, \varphi, t) \in (\mathbb{C}/\mathbb{Z}) \times (\mathbb{C}/\mathbb{Z}) \times (\mathbb{C}/\mathbb{Z}) : \|\operatorname{Re}(q, \varphi, t)\| < \sigma\}$$

and the space $\mathcal{P}_\sigma$ of real analytic functions on Q extensible to a bounded function on $\mathcal{D}_\sigma$. Identifying a function $f \in \mathcal{P}_\sigma$ with its trivial extension to $\mathcal{B}_\sigma(M)$, $\mathcal{P}_\sigma \subset \mathcal{B}_\sigma(U)$ for any U.

Theorem 2.1. *Let $U \subset M$ be open. Let $I^- < I^+$ be such that*

$$\{(p_0(t), q_0(t)) : t \in \mathbb{R}\} \times [I^-, I^+] \times \mathbb{T} \times \mathbb{T} \subset U.$$

Then for any $h \in \mathcal{A}_\sigma(U)$, there exists an open and dense set $\mathcal{G}_h$ of $\mathcal{P}_\sigma$ such that, for any $g \in \mathcal{G}_h$, the following holds for the Hamiltonian system

$$H_\epsilon = \frac{1}{2}p^2 + V(q) + G(I) + \epsilon\,(h(p, q, I, \varphi, t) + g(q, \varphi, t)).$$

There exists $\epsilon_0(h+g) > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, the system H_ϵ admits a trajectory $(p, q, I, \varphi)(t)$, $t \in [0, T]$, satisfying

$$I(0) < I^-, \quad I(T) > I^+.$$

Remark 1. Our theorem says that for the system $\frac{1}{2}p^2 + V(q) + G(I) + \epsilon h$ a diffusion trajectory whose I variable varies over the range $[I^-, I^+]$ always exists after a (possibly arbitrarily small) generic potential perturbation ϵg. This is commonly referred to as *generic in the sense of Mañé* (see [28]), which is stronger than genericity in the space $\mathcal{A}_\sigma$. The size of diffusion is independent of the size of the function $G(I)$, which may be relevant to multi-time-scale systems.

Remark 2. Let $\mathcal{X}$ be a subspace of the Banach space $\mathcal{A}_\sigma$. We say a subset $\mathcal{G} \subset \mathcal{A}_\sigma$ is generic along transversals $\mathcal{X}$ if for any $h \in \mathcal{A}_\sigma$, the set $\{g \in \mathcal{X} : h + g \in \mathcal{G}\}$ is open and dense in $\mathcal{X}$. The concept of Mañé genericity is then a special case when the transversals are chosen to be the space of all potentials.

3 Geometric Description of Diffusion for a Generic Perturbation

3.1 Scattering map and the geometric construction of diffusion For $\varepsilon = 0$, Hamiltonian (4) becomes

$$\frac{1}{2}p^2 + V(q) + G(I),$$

with associated equations

$$\dot{p} = V'(q), \quad \dot{q} = p, \qquad \dot{I} = 0, \quad \dot{\varphi} = \omega(I), \quad \dot{s} = 1,$$

so that I is a constant of motion and the flow based on the homoclinic trajectory (5) has the form

$$\Phi_0^t(I, \varphi) = (p_0(t), q_0(t), I, \varphi + t\omega(I)).$$

For any $I \in \mathbb{R}$, $\widetilde{\mathcal{T}}_I^0 = \left\{(0, 0, I, \varphi, s); (\varphi, s) \in \mathbb{T}^2\right\}$ is an invariant 2D-torus under the flow of the system with frequency $\tilde{\omega}(I) = (\omega(I), 1)$ and is called a *whiskered torus*. For each whiskered torus $\tilde{\mathcal{T}}_I^0$, we have associated coincident stable and unstable 3D-manifolds called *whiskers*, which we denote by

$$W^0\widetilde{\mathcal{T}}_I^0 = \left\{(p_0(\tau), q_0(\tau), I, \varphi, s) : \tau \in \mathbb{R}, (\varphi, s) \in \mathbb{T}^2\right\}.$$

The union of all whiskered tori $\widetilde{\mathcal{T}}_I^0$

$$\tilde{\Lambda}_0 = \left\{(0, 0, I, \varphi, s) : (I, \varphi, s) \in \mathbb{R} \times \mathbb{T}^2\right\}$$

is a 3D-*Normally Hyperbolic Invariant Manifold* (NHIM) with 4D-coincident stable and unstable invariant manifolds, forming a so-called *separatrix*, given by

$$W^0\tilde{\Lambda}_0 = \{(p_0(\tau), q_0(\tau), I, \varphi, s) : \tau \in \mathbb{R}, (I, \varphi, s) \in \mathbb{R} \times \mathbb{T}^2\}.$$

For $0 < \varepsilon \ll 1$, the NHIM $\tilde{\Lambda}_0$ is preserved to a NHIM $\tilde{\Lambda}_\varepsilon$ [13, Sec. 4.2], and the separatrix $W^0\tilde{\Lambda}_0$ is locally preserved to the stable manifold $W^{\mathrm{s}}(\tilde{\Lambda}_\varepsilon)$ and unstable manifold $W^{\mathrm{s}}(\tilde{\Lambda}_\varepsilon)$ of $\tilde{\Lambda}_\varepsilon$, which are defined as

$$W^{\mathrm{s,u}}(\tilde{\Lambda}_\varepsilon) = \bigcup_{x \in \tilde{\Lambda}_\varepsilon} W_x^{\mathrm{s,u}}(\tilde{\Lambda}_\varepsilon), \quad W_x^{\mathrm{s,u}}(\tilde{\Lambda}_\varepsilon) = \left\{z : \mathrm{dist}\left(\Phi_\varepsilon^t(z), \Phi_\varepsilon^t(x)\right) \underset{t \to \pm\infty}{\longrightarrow} 0\right\}.$$

In general, these two invariant manifolds no longer coincide, that is, the separatrix splits. The existence of this splitting can be detected by a perturbation argument in terms of the *Melnikov potential*

$$\mathcal{L}(I, \varphi, s) = \mathcal{L}_h(I, \varphi, s) := -\int_{-\infty}^{\infty} \big[h(p_0(t), q_0(t), \varphi + t\,\omega(I), I, s + t) \\ - h(0, 0, \varphi + t\,\omega(I), I, s + t)\big]dt. \tag{6}$$

Notice that this integral is absolutely convergent, because the homoclinic trajectory (5) converges exponentially fast to $(0, 0)$ as $t \to \pm\infty$.

Proposition 3.1 (DLS06). *Given* $(I, \varphi, s) \in \left[-I^*, I^*\right] \times \mathbb{T}^2$*, assume that the real function*

$$\tau \in \mathbb{R} \longmapsto \mathcal{L}(I, \varphi - \tau\, \omega(I),\ s - \tau) \in \mathbb{R} \tag{7}$$

has a non-degenerate critical point $\tau^* = \tau^*(I, \varphi, s)$*. Then, for* $0 < \varepsilon$ *small enough, there exists a unique transverse homoclinic point* $\tilde{z}_\varepsilon$ *to* $\tilde{\Lambda}_\varepsilon$ *of Hamiltonian* (4)*, which is* ε*-close to the point* $\tilde{z}^*(I, \varphi, s) = (p_0(\tau^*), q_0(\tau^*), I, \varphi, s) \in W^0(\tilde{\Lambda})$*:*

$$\tilde{z}_\varepsilon = \tilde{z}_\varepsilon(I, \varphi, s) = (p_0(\tau^*) + O(\varepsilon), q_0(\tau^*) + O(\varepsilon), I, \varphi, s)$$
$$\text{and } \tilde{z}_\varepsilon \in W^u(\tilde{\Lambda}_\varepsilon) \pitchfork W^s(\tilde{\Lambda}_\varepsilon).$$

Using this Proposition we can introduce the notion of the *scattering map*, which plays a central rôle in our mechanism for detecting diffusion. Let W be an open set of $\left[-I^*, I^*\right] \times \mathbb{T}^2$ such that the invariant manifolds of the NHIM $\tilde{\Lambda}_\varepsilon$ intersect transversely along a homoclinic manifold $\Gamma_\varepsilon = \{\tilde{z}_\varepsilon(I, \varphi, s), (I, \varphi, s) \in W\}$ and for any $\tilde{z}_\varepsilon \in \Gamma_\varepsilon$ there exists a unique $\tilde{x}_\pm = \tilde{x}_{\pm,\varepsilon}(I, \varphi, s) \in \tilde{\Lambda}_\varepsilon$ such that $\tilde{z}_\varepsilon \in W^s_\varepsilon(\tilde{x}_-) \cap W^u_\varepsilon(\tilde{x}_+)$. Let

$$H_\pm = \bigcup \left\{\tilde{x}_\pm = \tilde{x}_{\pm,\varepsilon}(I, \varphi, s) : (I, \varphi, s) \in W\right\}.$$

The scattering map associated to Γ_ϵ is the map

$$\begin{aligned} S_\varepsilon : H_- &\longrightarrow H_+ \\ \tilde{x}_- &\longmapsto S_\varepsilon(\tilde{x}_-) = \tilde{x}_+. \end{aligned}$$

Notice that the domain of definition of the scattering map depends on the homoclinic manifold chosen, but it is a subset of $O(1)$ size in $\tilde{\Lambda}_\varepsilon$ with respect to ε, i.e., it has a non-trivial limit as $\varepsilon \to 0$. If necessary, we can even restrict it further to avoid a non trivial monodromy in the definition of the scattering map, see, for instance, the comments in [11]. Therefore, for the characterization of the scattering maps, it is required to select the homoclinic manifold Γ_ϵ, which is determined by the function $\tau^*(I, \varphi, s)$. Once a function $\tau^*(I, \varphi, s)$ is chosen, by the geometric properties of the scattering map, particularly its exactness, see [13], the scattering map $S_\varepsilon = S_{\varepsilon,\tau^*}$ has the explicit form [12, eq. (9.9)]

$$S_\varepsilon(I, \varphi, s) = \left(I + \varepsilon \partial_\varphi L^* + \mathcal{O}(\varepsilon^2), \varphi - \varepsilon \partial_I L^* + \mathcal{O}(\varepsilon^2), s\right), \tag{8}$$

where $L^* = L^*(I, \varphi, s)$ is the *Poincaré function defined by*

$$L^*(I, \varphi, s) = \mathcal{L}\left(I, \varphi - \tau^*(I, \varphi, s)\, \omega(I), s - \tau^*(I, \varphi, s)\right). \tag{9}$$

Notice that if $\tau^*(I, \varphi, s)$ is a critical point of (7), $\tau^*(I, \varphi, s) - \sigma$ is a critical point of

$$\begin{aligned} \tau \in \mathbb{R} \longmapsto &\mathcal{L}(I, \varphi - (\tau + \sigma)\, \omega(I),\ s - (\tau + \sigma)) \\ &= \mathcal{L}(I, \varphi - \sigma\, \omega(I) - \tau\, \omega(I), s - \sigma - \tau) \end{aligned} \tag{10}$$

Since $\tau^*(I, \varphi - \sigma\, \omega(I), s - \sigma)$ is a critical point of the right-hand side of (10), by the uniqueness in W we can conclude that

$$\tau^*(I, \varphi - \sigma\, \omega(I), s - \sigma) = \tau^*(I, \varphi, s) - \sigma.$$

Thus, by (9), the Poincaré function L^* satisfies

$$L^*(I, \varphi - \sigma\, \omega(I), s - \sigma) = L^*(I, \varphi, s), \quad \sigma \in \mathbb{R},$$

and, in particular, for $\sigma = s$,

$$L^*(I, \varphi - s\,\omega(I), 0) = L^*(I, \varphi, s).$$

In other words, the Poincaré function L^* is invariant under the flow $\dot\varphi = \omega(I), \dot s = 1$ or, equivalently, is constant along any line in the (φ, s)-plane $\varphi - s\,\omega(I) =$ constant.

These previous equations are telling us that we can reduce the variables in both the Poincaré function L^* and the time τ^*. Indeed, writing them for $\sigma = s$ and introducing the variable

$$\theta = \varphi - s\,\omega(I), \tag{11}$$

we can define the *reduced Poincaré function* $\mathcal{L}^*$ as well as the reduced time $\bar\tau^*$, defined only in the variables (I, θ), as

$$\mathcal{L}^*(I, \theta) := L^*(I, \varphi - s\,\omega(I), 0) = L^*(I, \varphi, s),$$

$$\bar\tau^*(I, \theta) := \tau^*(I, \varphi - s\,\omega(I), 0) = \tau^*(I, \varphi, s) - s. \tag{12}$$

Moreover we also have

$$\mathcal{L}^*(I, \theta) = \mathcal{L}(I, \theta - \bar\tau^*(I, \theta)\,\omega(I), -\bar\tau^*(I, \theta)). \tag{13}$$

Along this paper, both $\tau^*(I, \varphi, s)$ and $\bar\tau^*(I, \theta)$ will be used at our convenience.

Note that the variable s is fixed under the scattering map (8). As a consequence, we can consider, for instance, the section

$$\Lambda_\epsilon = \tilde\Lambda_\epsilon \cap \{s = 0\}$$

and the scattering map S_ϵ is well defined on Λ_ϵ.

In the variables (I, θ) introduced in (11), the scattering map has the simple form

$$\mathcal{S}_\varepsilon(I, \theta) = \mathcal{S}_\varepsilon(I, \varphi) = \left(I + \varepsilon \frac{\partial \mathcal{L}^*}{\partial \theta}(I, \theta) + \mathcal{O}(\varepsilon^2), \theta - \varepsilon \frac{\partial \mathcal{L}^*}{\partial I}(I, \theta) + \mathcal{O}(\varepsilon^2)\right).$$

So, up to $\mathcal{O}(\varepsilon^2)$ terms, $\mathcal{S}_\varepsilon(I, \theta)$ is the $-\varepsilon$ times flow of *autonomous* Hamiltonian $\mathcal{L}^*(I, \theta)$. In particular, a finite number of iterates under the scattering map follow the level curves of $\mathcal{L}^*$ up to $\mathcal{O}(\varepsilon^2)$.

Remark 3. $\mathcal{S}_\epsilon$ is close to identity only when there is no phase shift, i.e., thanks to the fact that we are only dealing with an *uncoupled* a priori unstable system. If we introduce a coupling term between p and I in Hamiltonian (4), then one might expect that

$$\mathcal{S}_\epsilon(I, \theta) = (I, \theta + \gamma(I)) + \mathcal{O}(\epsilon),$$

where $\gamma(I)$ is the phase shift. In this case $\mathcal{S}_\epsilon$ is no longer close to identity.

We now introduce another map defined on the NHIM Λ_ϵ. Since $\tilde\Lambda_\varepsilon$ is invariant under the Hamiltonian flow and Λ_ϵ is a global section, there exists a Poincaré map

$$T_\epsilon : \Lambda_\epsilon \to \Lambda_\epsilon.$$

The map T_ϵ preserves the restriction of the symplectic form to Λ_ϵ, and has the expansion

$$T_\epsilon(I, \varphi) = (I, \varphi + \omega(I)) + \mathcal{O}(\varepsilon).$$

We call this map the *inner map*.

In the sequel, we call a measure on a manifold a *smooth measure* if it is given by a non-zero density multiplied by the volume form supported on an open set.

Lemma 3.2. *There exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ the following hold. There exists $\gamma_-, \gamma_+ \subset \Lambda_\epsilon$ that are graphs over the variable θ and invariant under T_ϵ, such that the T_ϵ-invariant subset of Λ_ϵ bounded by γ_- and γ_+ contains*

$$[I^-, I^+] \times \mathbb{T} \cap \Lambda_\epsilon.$$

In particular, the dynamics of T_ϵ on Λ_ϵ preserves a smooth measure whose support contains $[I^-, I^+] \times \mathbb{T} \cap \Lambda_\epsilon$.

We now restrict Λ_ϵ to the compact invariant set between $\gamma_\pm$, which is still called Λ_ϵ.

The following statement is a minor modification of the main result of [24], and is our main technical tool for construction of diffusion orbits. This result is non-perturbative, so we formulate it for a general Hamiltonian H with a compact invariant NHIM Λ.

Proposition 3.3. *Assume the Hamiltonian H admits a compact NHIM Λ on which multiple scattering maps $\mathcal{S}_j$ are defined on open subsets of Λ. Assume that the inner map T preserves a smooth measure on Λ, and each $\mathcal{S}_j$ maps positive (Lebesgue) measure sets to positive measure sets and zero measure sets to zero measure sets.*

Let $x_0, \ldots, x_{N-1}$ be a pseudo-orbit [1] *under the iterate of either the inner map or one of the scattering maps, that is,*

$$x_{i+1} = f_i(x_i), \quad 1 \le i \le N-1,$$

where f_i is either T or one of the scattering maps S_j. Then for any $\delta > 0$, there exists a trajectory $z : \mathbb{R} \to M$ of H_ϵ shadowing the pseudo-orbit x_i, i.e., there exists $t_0 < \cdots < t_N$ such that

$$\operatorname{dist}(x_i, z(t_i)) < \delta, \quad 0 \le i \le N.$$

Theorem 3.7 of [24] states that if the pseudo-orbit only involves the scattering maps, then the shadowing result holds. However, the proof in [24] can be adapted to prove Proposition 3.3 with minor changes, see Appendix A.

Remark 4. Proposition 3.3 does not provide an estimate of "diffusion time", i.e., upper bound estimates on t_i. This is due to the lack of return time estimates for the inner dynamics. See Remark 3.15 of [24].

3.2 Conditions on the Melnikov potential

Definition 2. Define $\mathcal{G}(I^-, I^+)$ to be the set of functions h in the space $\mathcal{A}_\sigma(U)$ introduced in Def. 1 for which the following holds:

- There exists intervals (I_j^-, I_j^+), $j = 1, \ldots, k$ such that

$$I_1^- < \cdots < I_k^-, \quad (I_j^-, I_j^+) \cap (I_{j+1}^-, I_{j+1}^+) \neq \emptyset, \quad j = 1, \ldots, k-1,$$

  and

$$[I^-, I^+] \subset \bigcup_j (I_j^-, I_j^+).$$

[1] We use the term pseudo-orbit to stress that it is not an orbit of the Hamiltonian dynamics, even though it is an orbit of the polysystem (i.e. iteration under multiple maps).

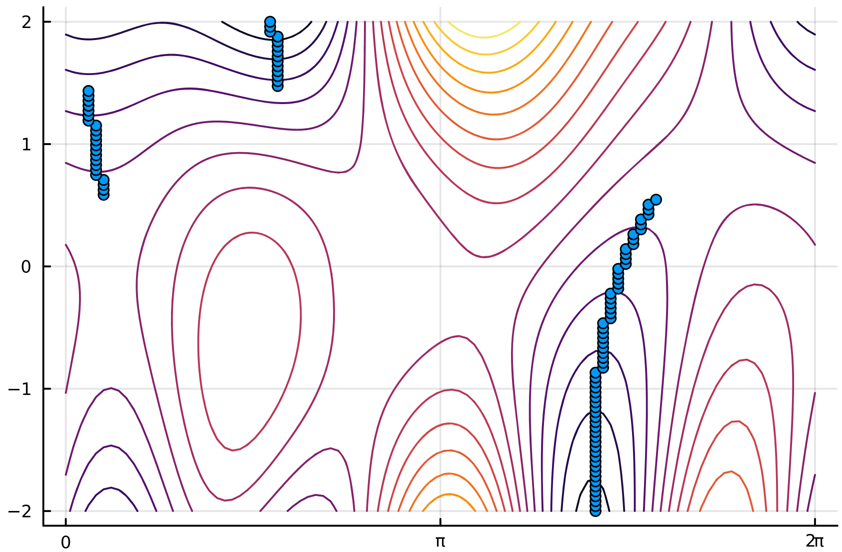

Fig. 1. The levels curves and global minima of a typical reduced Poincaré function $\mathcal{L}_h^*(I, \theta)$ defined on $\mathbb{R} \times \mathbb{T}$. Note: the vertical coordinate is I and horizontal coordinate is θ

- For each j, there exists a smooth mapping

$$(\varphi_j^*, s_j^*) : [I_j^-, I_j^+] \to \mathbb{T} \times \mathbb{T},$$

such that for each $I \in [I_j^-, I_j^+]$, $(\varphi_j^*, s_j^*)(I)$ is a non-degenerate local minimum of the function

$$\mathcal{L}_h(I, \cdot, \cdot).$$

We will show (Proposition 3.4) that for a generic h, the associated Melnikov potential $\mathcal{L}_h(I, \varphi, s)$ satisfies the conditions in Definition 2. In fact, the local minima (φ_j^*, s_j^*) can be chosen as local extension of the global minima of $\mathcal{L}_h(I, \cdot, \cdot)$. Moreover, in Lemma 3.7 we will see that they generate local minima $\theta_j^*(I)$ of $\mathcal{L}_h^*(I, \cdot)$. In Figure 1, we give an illustration of the position of global minima over $\mathbb{T}$ for a typical reduced Poincaré function $\mathcal{L}_h^*(I, \theta)$.

Proposition 3.4. *For any $h \in \mathcal{A}_\sigma(U)$, there exists an open and dense set $\mathcal{G}_h \subset \mathcal{P}_\sigma$ such that $h + g \in \mathcal{G}(I^-, I^+)$ for all $g \in \mathcal{G}_h$.*

Proposition 3.5. *Suppose $h \in \mathcal{G}(I^-, I^+)$, then there exists $\epsilon_0(h) > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, there exists an orbit $(p, q, I, \varphi)(t)$ of (4) and $T > 0$ such that*

$$I(0) < I^- < I^+ < I(T).$$

Proposition 3.4 will be proven later. In the sequel, let us fix $h \in \mathcal{G}(I^-, I^+)$ and prove existence of a diffusion trajectory.

Lemma 3.6. *Suppose $h \in \mathcal{G}(I^-, I^+)$, then there exists neighborhoods $U_j(I)$ of $(\varphi_j^*(I), s_j^*(I))$ for $I \in [I_j^-, I_j^+]$, and functions $\tau_j^*(I, \varphi, s)$*

$$(\varphi, s) \in U_j(I) \mapsto \tau_j^*(I, \varphi, s), \quad \tau_j^*(I, \varphi_j^*(I), s_j^*(I)) = 0,$$

depending smoothly in (φ, s, I), such that each τ_j^* is a non-degenerate local minimum of

$$\tau \mapsto \mathcal{L}_h(I, \varphi - \tau\omega(I), s - \tau). \tag{14}$$

Proof. We need to solve the equation

$$\partial_\tau \left(\mathcal{L}_h(I, \varphi - \tau\omega(I), s - \tau)\right) = -\left(\omega(I) \cdot \partial_\varphi \mathcal{L}_h + \partial_s \mathcal{L}_h\right)|_{(I, \varphi - \tau\omega(I), s - \tau)} = 0. \tag{15}$$

Since $(\varphi_j^*(I), s_j^*(I))$ is a non-degenerate local minimum of $\mathcal{L}_h(I, \cdot, \cdot)$, $\tau = 0$ solves (15) at $(\varphi_j^*(I), s_j^*(I))$.

Since $\partial^2_{(\varphi,s)}\mathcal{L}_h(I, \varphi_j^*(I), s_j^*(I))$ is positive definite, and

$$\begin{aligned}
&\partial_\tau \left(-\left(\omega(I) \cdot \partial_\varphi \mathcal{L}_h + \partial_s \mathcal{L}_h\right)|_{(I, \varphi - \tau\omega(I), s - \tau)}\right) \\
&= -\partial_\tau \left(\partial_{(\varphi,s)}\mathcal{L}_h \begin{bmatrix} \omega(I) \\ 1 \end{bmatrix} |_{(I, \varphi - \tau\omega(I), s - \tau)}\right) \\
&= \left[\omega(I)\ 1\right] \partial^2_{(\varphi,s)}\mathcal{L}_h(I, \varphi - \tau\omega(I), s - \tau) \begin{bmatrix} \omega(I) \\ 1 \end{bmatrix},
\end{aligned}$$

we get

$$\partial_\tau^2 \left(\mathcal{L}_h(I, \varphi_j^*(I) - \tau\omega(I), s_j^*(I) - \tau)\right) > 0. \tag{16}$$

By the implicit function theorem, there exists a unique $\tau_j^*(I, \varphi, s)$, depending smoothly on (I, φ, s) in a neighborhood of the graph $(I, \varphi_j^*(I), s_j^*(I))$ over the variable I, solving equation (15). Moreover, τ_j^* are non-degenerate local minima for the mapping (14) due to (16). □

In view of this Lemma, on each interval (I_j^-, I_j^+) and for $\theta = \varphi - s\,\omega(I)$ with $(\varphi, s) \in U_j(I)$, we can now introduce $\bar{\tau}_j^*(I, \theta)$ and $\mathcal{L}_j^*(I, \theta)$ by the same formulas (12,13) defining $\bar{\tau}^*(I, \theta)$ and $\mathcal{L}^*(I, \theta)$, simply changing τ^* to τ_j^*. We will also denote

$$\theta_j^*(I) = \varphi_j^*(I) - s_j^*\omega(I).$$

Lemma 3.7. *For each $I \in [I_j^-, I_j^+]$, $\theta_j^*(I)$ is a non-degenerate minimum of $\mathcal{L}_j^*(I, \cdot)$.*

Proof. If $\tau^* = \tau^*(I, \varphi, s)$ is a critical point of (14), it satisfies

$$\left(\omega(I)\partial_\varphi \mathcal{L}_h + \partial_s \mathcal{L}_h\right)\left(I, \varphi - \tau^*\omega(I), s - \tau^*\right) = 0,$$

so that we also have

$$\begin{aligned}
&\left(\omega(I)\partial_\varphi \mathcal{L}_h + \partial_s \mathcal{L}_h\right)\Big|_{(I, \theta - \bar{\tau}^*(I,\theta)\,\omega(I), -\bar{\tau}^*(I,\theta))} \\
&= \left(\omega(I)\partial_\varphi \mathcal{L}_h + \partial_s \mathcal{L}_h\right)\Big|_{(I, \theta - \tau^*(I,\theta,0)\,\omega(I), -\tau^*(I,\theta,0))} = 0.
\end{aligned} \tag{17}$$

Denote by id the identity matrix, then differentiating (13) we get

$$\begin{aligned}\partial_\theta \mathcal{L}_j^*(I,\theta) &= \partial_\varphi \mathcal{L}_h \left(\mathrm{id} - \partial_\theta \bar\tau_j^*\, \omega(I) - \partial_\theta \bar\tau_j^* \partial_s \mathcal{L}_h\right)\Big|_{(I,\theta-\bar\tau_j^*\,\omega(I),-\bar\tau_j^*)} \\ &= \left(\partial_\varphi \mathcal{L}_h - \partial_\theta \bar\tau_j^* \left(\omega(I)\partial_\varphi \mathcal{L}_h + \partial_s \mathcal{L}_h\right)\right)\Big|_{\left(I,\theta-\bar\tau_j^*\omega(I),-\bar\tau_j^*\right)} \\ &= \partial_\varphi \mathcal{L}_h\Big|_{(I,\theta-\bar\tau_j^*\,\omega(I),-\bar\tau_j^*)}\end{aligned}$$

by (17). Since $\bar\tau_j^* = \bar\tau_j^*(I,\theta_j^*) = \tau_j^*(I,\varphi_j^*(I), s_j^*(I)) = -s_j^*(I)$ by (12), we have that $(I,\theta - \bar\tau_j^*\,\omega(I), -\bar\tau_j^*) = (I,\varphi_j^*(I), s_j^*(I))$, and we get that $\theta_j^*(I)$ is a critical point of $\mathcal{L}_j^*(I,\cdot)$. Furthermore, repeating the same calculation, we have

$$\partial^2_{\theta\theta}\mathcal{L}_j^* = \partial^2_{\varphi\varphi}\mathcal{L}_h\Big|_{(I,\theta-\bar\tau_j^*\,\omega(I),-\bar\tau_j^*)}.$$

Since $\partial^2_{\varphi\varphi}\mathcal{L}_h$ is positive definite at (φ_j^*, s_j^*), θ_j^* is a non-degenerate local minimum. □

Lemma 3.8 (Ascending ladder). *There exists* $\delta_0 > 0$ *such that for every* $\delta \in (0,\delta_0)$*, there exists a sequence of curves*

$$\gamma_{i,j} \subset [I^-, I^+] \times \mathbb{T} \quad 1 \le i \le m_j,\ 1 \le j \le k,\ l \in \{1,2\}$$

such that the following holds.

(1) For each i, j*, the curve* $\gamma_{i,j}$ *is a smooth graph over the variable* I*, i.e.,*

$$\gamma_{i,j} = \{(I, f_{i,j}(I)) :\ I \in [I^1_{i,j}, I^2_{i,j}]\}.$$

$\gamma_{i,j}$ is a segment of the level curve of $\mathcal{L}_j^*$ and the Hamiltonian flow of $\mathcal{L}_j^*$ on $\gamma_{i,j}$ is increasing in the I component.

(2) $I^1_{1,j} \in [I_j^-, I_j^- + \delta]$, $I^2_{m_j,j} \in [I_j^+ - \delta, I_j^+]$.

(3) $I^2_{m_j,j} = I^1_{1,j+1}$ *for all* $j = 1,\dots,k-1$.

Let us also denote by $x^1_{i,j}, x^2_{i,j}$ the lower and upper end points of the curves $\gamma_{i,j}$.

Proof. Since $\theta_j^*(I)$ is a non-degenerate local minimum of $\partial_\theta\mathcal{L}_j^*(I,\cdot)$, for every $I \in [I_j^-, I_j^+]$, there exists $\phi(I) > \theta_j^*(I)$, such that

$$\partial_\theta \mathcal{L}_j^*(I,\phi(I)) > 0.$$

This means there is a segment $\gamma_j(I)$ of the level curve of $\mathcal{L}_j^*$, on which I is increasing under the Hamiltonian flow of $\mathcal{L}_j^*$. By compactness, there exists a finite collection $\gamma_{i,j}$ of such level curves, $1 \le i \le m_j$ such that $\bigcup_i \pi_I(\gamma_{i,j}) \supset [I_j^-, I_j^+]$. One can adjust the collection so that $\pi_I(\gamma_{i,j}) \cap \pi_I(\gamma_{i,j+1}) \neq \emptyset$ and $I_j^- \in \pi_I(\gamma_{1,j})$ and $I_j^+ \in \pi_I(\gamma_{m_j,j})$. It suffices to choose

$$\begin{aligned} I^1_{i,j} &\in \pi_I(\gamma_{i,j}) \cap \pi_I(\gamma_{i,j-1}), \quad j \ge 2, \\ I^2_{i,j} &\in \pi_I(\gamma_{i,j}) \cap \pi_I(\gamma_{i,j+1}), \quad j \le m_j + 1,\end{aligned}$$

and $I^1_{1,j}$, $I^2_{2,j}$ satisfying (2) and (3); then truncate the curves $\gamma_{i,j}$ to the I interval $[I^1_{i,j}, I^2_{i,j}]$. □

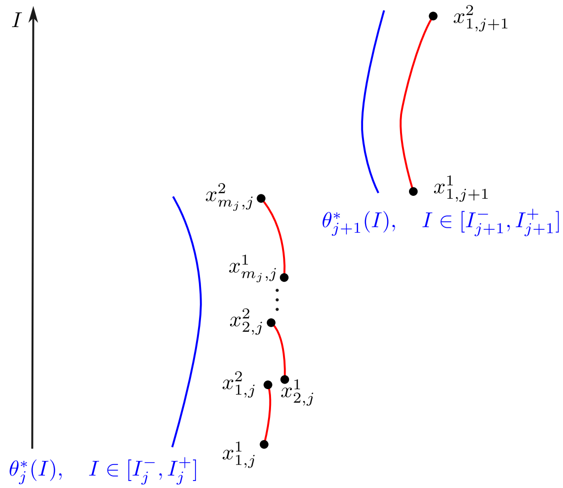

Fig. 2. The ascending ladder

3.3 Constructing the transition chain

Proposition 3.9 (Transition chain). *Let $h \in \mathcal{G}(I^-, I^+)$, and let $\gamma_{i,j}$ be the ascending ladder constructed in Lemma 3.8. Then there is $\delta_1 > 0$ such that for every $\delta \in (0, \delta_1)$, there exists $\epsilon_0 > 0$ depending on h and δ, such that for every $\epsilon \in (0, \epsilon_0)$, there exist*

$$y^l_{i,j} \in [I^-, I^+] \times \mathbb{T}, \qquad 1 \le i \le m_j,\ 1 \le j \le k,\ l \in \{1, 2\},$$

and $M_{i,j}, N_{i,j} \in \mathbb{N}$ such that the following hold (Figure 2).

Let $S_{j,\epsilon}$ denote the scattering map associated to $\mathcal{L}^*_j$, and T_ϵ denote the inner dynamics.

(1) $S^{M_{i,j}}_{j,\epsilon}(y^1_{i,j}) = y^2_{i,j}$.
(2) $T^{N_{i,j}}_\epsilon(y^2_{i,j}) = y^1_{i+1,j}$, $i = 1, \ldots, m_j - 1$.
(3) $T^{N_{m_j,j}}_\epsilon(x^2_{m_j,j}) = x^1_{1,j+1}$, $j = 1, \ldots, k-1$.
(4) $|x^l_{i,j} - y^l_{i,j}| < \delta$ for all $i = 1, \ldots, m_j$, $j = 1, \ldots, k$, $l = 1, 2$.

We first prove Proposition 3.5 assuming Proposition 3.9.

Proof of Proposition 3.5. Let $I(\cdot)$ denote the projection $(I, \theta) \mapsto I$. By Lemma 3.8 and Proposition 3.9, there exists a pseudo-orbit $\{y^l_{i,j}\}$ of the maps $S_{j,\epsilon}$ and T_ϵ, such that $I(y^1_{1,1}) < I^-$ and $I(y^2_{m_k,k}) > I^+$. By Proposition 3.3, there exists an orbit $(q, p, I, \varphi)(t)$ of the original Hamiltonian system and $T > 0$ such that

$$I(0) < I^-, \quad I(T) > I^+.$$

□

The remaining section is dedicated to proving Proposition 3.9.

We have the following simple lemma about integrable twist maps.

Lemma 3.10. *Consider the map* $T : [I^-, I^+] \times \mathbb{T} \to \mathbb{R} \times \mathbb{T}$ *defined by*

$$T(I, \theta) = (I, \theta + \omega(I)).$$

Assume that for every interval $(I_1, I_2) \subset [I^-, I^+]$*,* $\omega((I_1, I_2))$ *contains a nontrivial interval. Let* $T_\epsilon : [I^-, I^+] \times \mathbb{T} \to \mathbb{R} \times \mathbb{T}$ *be a family of maps such that*

$$\|T_\epsilon - T\|_{C_1} \leq C\epsilon$$

for an independent constant C*.*

Then for any two C^1 *graphs* γ_1*,* γ_2 *over the variable* I

$$\gamma_1 = \{(I, f(I)) : I \in (I_1, I_2)\}, \quad \gamma_2 = \{(I, g(I)) : I \in (I_1, I_2)\},$$

$x_1 \in \gamma_1$ *and* $\delta > 0$*, there exists* $\epsilon_0 > 0$*,* $N \in \mathbb{N}$*,* $x_2 \in \gamma_2$*,* $\delta' > 0$ *depending on* x_1*,* δ *depending only on* T *and* C*, and* I_1*,* I_2*, such that for all* $\epsilon \in (0, \epsilon_0)$*,*

$$T_\epsilon^N(B_\delta(x_1)) \supset B_{\delta'}(x_2).$$

Proof. By assumption, $\omega((I_1, I_2))$ contains a nontrivial interval (a, b). Let $c_1 = \inf_{I \in (I_1, I_2)} (f(I) - g(I))$ and $c_2 = \sup_{I \in (I_1, I_2)} (f(I) - g(I))$, then

$$\{f(I) - g(I) + N\omega(I) : I \in (I_1, I_2)\} \supset (Na + c_2, Nb + c_1). \tag{18}$$

Let N be such that $N(b - a) + c_1 - c_2 > 1$, then the set (18) projects onto the torus $\mathbb{T}$ in the θ component. This implies

$$T^N(\gamma) = \{f(I) + \omega(I) : I \in (I_1, I_2)\} \cap \gamma_2 \neq \emptyset$$

as subsets of $\mathbb{R} \times \mathbb{T}$. Denote the intersection $x_2 = T^N(x_1)$ where $x_1 \in \gamma_1$, $x_2 \in \gamma_2$.

It follows that there exists $\delta_1 > 0$ depending only on T and N such that

$$T^N(B_\delta(x_1)) \supset B_{\delta_1}(x_2).$$

Moreover, we can choose ϵ_0 small enough such that for all $\|T_\epsilon - T\|_{C^1} < C\epsilon_0$, we have

$$T_\epsilon^N(B_\delta(x_1)) \supset B_{\delta_1/2}(x_2).$$

□

Let $\gamma_{i,j}$, $1 \leq i \leq m_j$, $1 \leq j \leq k$, $l = 1, 2$ be the ascending ladder constructed in Lemma 3.8.

Lemma 3.11. *There exists* $\delta_0 > 0$ *such that for any* $\delta \in (0, \delta_0)$*, there exists* $\epsilon_0 > 0$ *and* $C > 1$ *depending only on the perturbation* h *and* δ*, such that for all* $1 \leq i \leq m_j$*,* $1 \leq j \leq k$*, any* $x_1, x_2 \in \gamma_{i,j}$ *with* $\pi_I(x_1) < \pi_I(x_2)$*, and* $\epsilon < \epsilon_0$ *there exists* $M \in \mathbb{N}$ *depending on* $x_1, x_2, \delta, \epsilon$ *such that*

$$S_{\epsilon,j}^M(B_\delta(x_1)) \supset B_{\delta/C}(x_2),$$

where $B_r(x)$ *denote the ball of radius* r *at* x*.*

Proof of Lemma 3.11. Let ϕ_j^t denote the Hamiltonian flow defined by the Hamiltonian $\mathcal{L}_j^*$. By construction, there exists $t > 0$ such that

$$\phi_j^t(x_1) = x_2$$

There exists $c > 0$ (depending only on $\mathcal{L}_j$, which depends only on h) and a local flow-box coordinate

$$\chi_{i,j} : (-c, t + c) \times (-c, c) \to \mathbb{T} \times \mathbb{R}$$

such that $x_1 = \chi_{i,j}(0, 0)$ and

$$\phi^t \circ \chi_{i,j}(s, m) = \chi_{i,j}(s + t, m).$$

In particular, $x_2 = \chi_{i,j}(0, t)$. Let $M = \lfloor t/\epsilon \rfloor$. Since $S_{\epsilon,j} = \phi_{i,j}^{\epsilon} + O(\epsilon^2)$, $S_{\epsilon,j}^M = \phi_{i,j}^{\epsilon t} + O(\epsilon)$,

$$\chi_{i,j}^{-1} \circ S_{\epsilon,j}^M \circ \chi_{i,j}(s, m) = (s + t, m) + O(\epsilon).$$

For any $r \in (0, c)$ and ϵ_0 small enough, we have

$$\chi_{i,j}^{-1} \circ S_{\epsilon,j}^M \circ \chi_{i,j}(B_r(s, m)) \supset B_{r/2}(s + t, m).$$

Let $C_1 = \max\left\{\sup \|D\chi_{i,j}\|, \sup \|D\chi^{-1}\|\right\}$ which depends only on the perturbation h. Then if $C_1\delta \in (0, c)$,

$$\begin{aligned} S_{\epsilon,j}^M(B_\delta(x_1)) &\supset \chi_{i,j} \circ \chi_{i,j}^{-1} \circ S_{\epsilon,j}^M \circ \chi_{i,j}(B_{\delta/C_1}(s, m)) \\ &\supset \chi_{i,j}(B_{\delta/(2C_1)}(s + t, m)) \supset B_{\delta/(2C_1^2)}(x_2). \end{aligned}$$

This conclude the proof of Lemma 3.11. □

Proof of Proposition 3.9. We now construct a sequence of balls

$$B_{\delta_{i,j}^l}(z_{i,j}^l) \subset B_\delta(x_{i,j}^l),$$

and $M_{i,j}, N_{i,j} \in \mathbb{N}$ such that

$$S_{\epsilon,j}^{M_{i,j}}(B_{\delta_{i,j}^1}(z_{i,j}^1)) \supset B_{\delta_{i,j}^2}(z_{i,j}^2), \quad 1 \le i \le m_j, \quad 1 \le j \le k, \tag{19}$$

and

$$T_\epsilon^{N_{i,j}}(B_{\delta_{i,j}^2}(z_{i,j}^2)) \supset B_{\delta_{i^*,j^*}^1}(z_{i^*,j^*}^1),$$

where $(i^*, j^*) = (i + 1, j)$ if $i < m_j$ and $(i^*, j^*) = (1, j + 1)$ if $i = m_j$ and $j < k$. This implies existence of $y_{i,j}^l \in B_{\delta_{i,j}^l}(z_{i,j}^l)$ satisfying the conclusion of our Proposition.

We choose $\delta_{1,1}^1 = \delta$ and $z_{1,1}^1 = x_{1,1}^1$. Constructing by induction, we assume $\delta_{i,j}^1$ and $z_{i,j}^1$ are already chosen. Choose $z_{i,j}^2 = x_{i,j}^2$ and apply Lemma 3.11 to get existence of $\epsilon_0 > 0$ and $M_{i,j}^\epsilon \in \mathbb{N}$ and $\delta_{i,j}^2$ such that (19) holds for all $\epsilon \in (0, \epsilon_0)$.

The curves $\gamma_{i,j}$ and γ_{i^*,j^*} are given by graphs over the variable I

$$\gamma_{i,j} = \{(I, f(I)) : I \in [I^1_{i,j}, I^2_{i,j}]\}, \quad \gamma_{i^*,j^*} = \{(I, g(I)) : I \in [I^1_{i^*,j^*}, I^2_{i^*,j^*}]\},$$

where $I^2_{i,j} = I^1_{i^*,j^*}$. Extending these curves along the level curves of $\mathcal{L}^*_j$ and $\mathcal{L}^*_{j^*}$, we may assume that the extended curves overlap in I direction for some interval

$$I \in [I^2_{i,j}, I^2_{i,j} + c].$$

Lemma 3.10 applies to the inner map. By possibly reducing ϵ_0, there exists $N \in \mathbb{N}$, $z^1_{i^*,j^*} \in \gamma_{i^*,j^*}$, and $\delta^1_{i^*,j^*} > 0$ such that for all $\epsilon \in (0, \epsilon_0)$,

$$T^N(B_{\delta^2_{i,j}}(z^2_{i,j})) \supset B_{\delta^1_{i^*,j^*}}(z^1_{i^*,j^*}).$$

We can continue the construction until all the balls are defined. Note that we need to reduce ϵ_0 up to finitely many times, but it will be well defined at the end of the construction. This concludes the proof of Proposition 3.9. □

4 Genericity

We prove Proposition 3.4 in this section. The proof follows a standard parametric transversality argument. In Section 4.1, we prove that the map $g \mapsto \mathcal{L}_{h+g}$ is satisfies a transversality condition. This transversality allows us to generically avoid certain degeneracies of derivatives.

4.1 Transversality of parameters Denote $x = (I, \varphi, s)$, and $X = \mathbb{R} \times \mathbb{T} \times \mathbb{T}$. The spaces of functions $\mathcal{P}_\sigma$ and $\mathcal{A}_\sigma$ are introduced in Def. 1.

Proposition 4.1. *At every $x \in \mathbb{R} \times \mathbb{T} \times \mathbb{T}$, the mapping*

$$E(g) := g \mapsto \begin{bmatrix} \partial_{(s,\varphi)} \\ \partial^2_{(s,\varphi)} \\ \partial^3_{(s,\varphi)} \end{bmatrix} \mathcal{L}_g(x), \quad \mathcal{P}_\sigma \to \mathbb{R}^{2+3+4} \tag{20}$$

is a surjection.

Before the proof, we state some Lemmas.

Lemma 4.2. *The mapping*

$$(x, h) \mapsto \mathcal{L}_h(x), \quad X \times \mathcal{A}_\sigma \to \mathbb{R}$$

is C^∞ in both x and h. Moreover, for every $l > r \geq 0$, the mapping $h \mapsto \mathcal{L}_h$ is well defined as a mapping $C^l \mapsto C^l$, and there exists a constant $M_r > 0$ such that

$$\|\mathcal{L}_{h_1} - \mathcal{L}_{h_2}\|_{C^r} \leq M_r \|h_1 - h_2\|_{C^{r+1}}.$$

Proof. Recall (6):

$$\mathcal{L}_h(I,\varphi,s) = -\int_{-\infty}^{\infty}\big[h(p_0(t),q_0(t),\varphi+\omega(I)t,I,s+t) - h(0,0,\varphi+\omega(I)t,I,s+t)\big]dt. \tag{21}$$

By assumption $(p_0(t),q_0(t))$ converges exponentially fast to $(0,0)$ as $t\to\pm\infty$, the integral (21) converges absolutely and uniformly over all $x\in X$, with the estimate

$$\sup_{x\in X}|\mathcal{L}_h(x)| \le C_r \sup_{x\in X}|\partial_{(q,p)}h| \tag{22}$$

for some $C_r>0$.

Therefore for any multi-index $\alpha\in\mathbb{N}^3$,

$$\partial_x^\alpha \mathcal{L}_h = \mathcal{L}_{\partial_x^\alpha h}.$$

Since h is analytic, $\mathcal{L}_h$ is C^∞ in x. $\mathcal{L}_h$ is C^∞ in h since it is linear in h.

Moreover, using (22), we have

$$\|\mathcal{L}_h\|_{C^r} = \sup_{|\alpha|\le r}\sup_x|\partial_x^\alpha\mathcal{L}_h| \le \sup_{|\alpha|\le r}\sup_x|\mathcal{L}_{\partial_x^\alpha h}| \le M_r\|h\|_{C^{r+1}}$$

for some $M_r>0$ depending on C_r. □

Given $r\ge 1$, let

$$\Delta_r = \{j,k\in\mathbb{Z}:\ j,k\ge 0,\ j+k\le r\}.$$

Lemma 4.3. *Given $r\ge 1$, $(I_0,s_0,\varphi_0)\in X$, and*

$$\{c_{j,k}\in\mathbb{R}:\ (j,k)\in\Delta_r\},$$

for every $\delta>0$, there exists $g\in\mathcal{P}_\sigma$ such that

$$|\partial_\phi^j\partial_s^k\mathcal{L}_g(I_0,s_0,\varphi_0) - c_{j,k}| < \delta,\quad \forall j,k\in\Delta_r.$$

Proof. Let $G_0(\varphi,s)$ be a smooth function that satisfies

$$\partial_\varphi^j\varphi_s^k G_0(\varphi_0,s_0) = c_{j,k},\quad \forall j,k\in\Delta_r.$$

In a neighborhood U of $q_0(0)$ we consider a coordinate change $\phi: U\to\mathbb{R}^d$ such that $\varphi(q_0(t)) = te_1$ for all t such that $q_0(t)\in U$, where e_1 is the first coordinate vector. Let $a>0$ be small enough such that $[-a,a]^d\subset\phi(U)$, and given $\delta_1>0$ let η_1 be a smooth function supported on $(-a/2,a/2)$ such that

$$\left|\int\eta_1(t)f(t)dt - f(0)\right| < \delta_1\|f\|_{C^r}$$

for any test function f. Let η be a smooth function such that $\eta(x_1,\dots,x_d)=\eta_1(x_1)$ on $[-a/2,a/2]^d$ and supported on $[-a,a]^d$, then define

$$R_0(q) = \eta\circ\varphi(q).$$

Let $f(t)$ be a smooth test function, we have

$$\left|\int_{-\infty}^{\infty} R_0(q_0(t))f(t)dt - f(0)\right| = \left|\int_{-\infty}^{\infty} \eta_1(te_1)f(t)dt - f(0)\right| \leq \delta_1\|f\|_{C^r}.$$

Define $g_0(q,\varphi,t) = R_0(q)G_0(\varphi,s)$, we get

$$\begin{aligned}&|\partial_\varphi^j\partial_s^k\mathcal{L}_g(I_0,\varphi_0,s_0) - c_{j,k}|\\ &= \left|\int_{-\infty}^{\infty}(R(q_0(t)) - R(0))\partial_\varphi^j\partial_s^k G(\varphi_0+\omega(I)t, s_0+t)dt - c_{j,k}\right|\\ &= \left|\int_{-\infty}^{\infty} R(q_0(t))\partial_\varphi^j\partial_s^k G(\varphi_0+\omega(I)t, s_0+t)dt - c_{j,k}\right| \leq \delta_1 C,\end{aligned}$$

where C is a constant that depends only on G. We now choose δ_1 so that $\delta_1 C < \delta/2$.

Since $\mathcal{P}_\sigma$ is dense in $C^{r+1}(\mathbb{T}^3)$, by Lemma 4.2, we can choose $g \in \mathcal{P}_\sigma$ such that

$$\|\mathcal{L}_g - \mathcal{L}_{g_0}\|_{C^r} < \delta/2.$$

This choice of g verifies the claim of our lemma. □

Proof of Proposition 4.1. We can represent the codomain of (20) by

$$\{c_{j,k} : j,k \in \Delta_r \setminus \{(0,0)\}\}.$$

Let $\mathbf{c}_1,\dots,\mathbf{c}_{10}$ be a basis be the above space, then by Lemma 4.3, for any $\delta > 0$, there exists $g_1,\dots,g_{10} \in \mathcal{P}_\sigma$ such that

$$\|E(g_i) - \mathbf{c}_i\|_\infty < \delta$$

for all $1 \leq i \leq 10$. It suffices to choose δ small enough so that $E(g_i)$ still form a basis. □

4.2 Generic property of critical points We proceed to prove Proposition 3.5.

Lemma 4.4. *There exists an open and dense subset $\mathcal{G}(I^-,I^+) \subset \mathcal{P}_\sigma$, such that for all $I \in [I^-,I^+]$ and $g \in \mathcal{G}(I^-,I^+)$, the function*

$$(s,\varphi) \mapsto \mathcal{L}_{h+g}(I,s,\varphi), \quad \mathbb{T}^2 \to \mathbb{R}$$

does not admit any degenerate local minima or maxima.

Proof. Let $\mathbb{S}^1 = \{v \in \mathbb{R}^2 : \|v\| = 1\}$, consider the mapping

$$F(x,v,g) = \begin{bmatrix}\partial_\varphi\mathcal{L}_{h+g}(x)\\ \partial_s\mathcal{L}_{h+g}(x)\\ \partial^2_{(s,\varphi)}\mathcal{L}_{h+g}(x)v\\ \partial^3_{(s,\varphi)}\mathcal{L}_{h+g}(x)[v,v,v]\end{bmatrix}, \quad X \times \mathbb{S}^1 \times \mathcal{P}_\sigma \to \mathbb{R}^5.$$

Since F is linear in g,

$$\partial_g F(x_0,v_0,g_0)(\delta g) = F(x_0,v_0,\delta g).$$

By Proposition 4.1, at every $(x_0, v_0, g_0) \in \mathbb{R} \times \mathbb{S}^1 \times \mathcal{P}_\sigma$, there exists a finite dimensional subspace $E \subset \mathcal{P}_\sigma$, such that the mapping

$$\delta g \mapsto \partial_g F(x_0, v_0, g_0)(\delta g) = F(x_0, v_0, \delta g), \quad E \to \mathbb{R}^5$$

is a surjection. Moreover, by the smoothness of the mapping F in (x_0, v_0, g_0), the same is true for all (x, v, g) contained in a neighborhood U_{loc} of $(x_0, v_0, g_0) \in X \times \mathbb{S}^1 \times E$. By the implicit function theorem, the set

$$W_{\mathrm{loc}} = \{(x, v, g) : F(x, v, g) = 0\}$$

is a submanifold of codimension 5 in U_{loc}. Let π_g be the projection into the g component, then by Sard's Theorem, the set of regular values $g \in \pi_g(U_{\mathrm{loc}})$, denoted $\mathcal{G}_{\mathrm{loc}}$, is a residual set in U_{loc}. Since $d\pi_g$ has at most co-rank 4, g_1 is a regular value implies $\pi_g^{-1}(g_1) \cap W_{\mathrm{loc}} = \emptyset$, which means $F(x, v, g_1) \neq 0$ for all $g_1 \in \mathcal{G}_{\mathrm{loc}}$ and $(x, v, g_1) \in U_{\mathrm{loc}}$.

Let U_i be a countable covering of the separable space $X \times \mathbb{S}^1 \times \mathcal{P}_\sigma$, we apply the above argument to each U_i to obtain a relatively residual set $\mathcal{G}_i \subset U_i$, such that $F(x, v, g_1) \neq 0$ for all $g_1 \in \mathcal{G}_i$ and $(x, v, g_1) \in U_i$. Then

$$\mathcal{G} = \bigcup_i \mathcal{G}_i$$

is a residual subset of $\mathcal{P}_\sigma$, such that for every $g \in \mathcal{G}$ and $(x, v) \in X \times \mathbb{S}^1$,

$$F(x, v, g) \neq 0.$$

Finally, set

$$\mathcal{G}(I^-, I^+) = \{g : F(x, v, g) \neq 0, \text{ for all } x, v \in [I^-, I^+] \times \mathbb{T} \times \mathbb{T} \times \mathbb{S}^1\},$$

this set is open due to compactness in (x, v). It is also dense since $\mathcal{G}(I^-, I^+) \supset \mathcal{G}$.

To prove our lemma, let $g \in \mathcal{G}(I^-, I^+)$ and let $x_0 = (I_0, s_0, \varphi_0)$ be a degenerate critical point of $\mathcal{L}_{h+g}$, i.e., $\partial_{(s,\varphi)} \mathcal{L}_{h+g} = 0$ and there exists $v \in \mathbb{S}^1$ such that

$$\partial^2_{(s,\varphi)} \mathcal{L}_{h+g}(x_0) v = 0.$$

Since $F(x, v, g) \neq 0$, we must have

$$\partial^3_s(s, \varphi) \mathcal{L}_{h+g}(x)[v, v, v] \neq 0.$$

We conclude that the function $t \mapsto \mathcal{L}_{h+g}(x_0 + t(0, v))$ is an open mapping near $t = 0$, hence (s_0, φ_0) cannot be a local minimum or maximum of $\mathcal{L}_{h+g}(I_0, \cdot, \cdot)$. □

Proof of Proposition 3.5. By Lemma 4.4, if $g \in \mathcal{G}(I^-, I^+)$, then for all $I \in [I^-, I^+]$ the function $\mathcal{L}_{h+g}(I, \cdot, \cdot)$ has no degenerate minima or maxima.

Fix $g \in \mathcal{G}(I^-, I^+)$, and let $M_g \subset \mathbb{T} \times \mathbb{T}$ denote the set of all global minima of $\mathcal{L}_{h+g}(I, \cdot, \cdot)$ for some $I \in [I^-, I^+]$. Since all the minima are non-degenerate, they are isolated on each level $\{I\} \times \mathbb{T} \times \mathbb{T}$. Moreover, any non-degenerate local minima can be extended smoothly to a neighborhood of I. We conclude that M_g is contained in a finite family of curves of non-degenerate local minima $\{(I, \varphi^*_J(I), s^*_J(I)) : I \in J\}$ where $J \in \mathcal{J}$ is a finite collection of open intervals, and $\bigcup_{J \in \mathcal{J}} J \supset [I^-, I^+]$. We now select the intervals $[I^-_j, I^+_j]$ inductively as such:

- Let (I_1^-, I_1^+) be any interval in $\mathcal{J}$ what contains I^-.
- If intervals (I_j^-, I_j^+) are selected for $j \leq m$, pick any interval (a, b) in $\mathcal{J}$ that contains I_m^+. Set $I_{m+1}^- = \max\{a, (I_m^- + I_m^+)/2\}$ and $I_{m+1}^+ = b$. □

Acknowledgements. AD is supported by Spanish grant PID2021-123968NB-I00 (MICIU/AEI/10.13039/501100011033/FEDER/UE). KZ is supported by NSERC fund RGPIN-2019-07057.

Data Availability The authors declare that the data supporting the findings of this study are available within the paper.

Declarations

Conflicts of Interest Both authors have contributed equally to the conception and design of the presented study, and declare no conflict of interest.

Appendix A. Shadowing Pseudo-Orbits Using Poincaré Recurrence

In this section we describe the proof of Proposition 3.3. This Proposition has minor differences from Theorem 3.7 of [24] by allowing the iterates of inner dynamics. Our assumption on the measure preserving properties of T and S_j is also stronger than that of [24] but nevertheless satisfied for a priori unstable systems. The proof follows the main ideas of [24].

Lemma A.1 (Lemma 3.2 of [24]). *Suppose Λ_ϵ is an NHIM of the Hamiltonian H and T is the associated inner map. Assume that there exists a finite family S_j, $1 \leq j \leq m$, of scattering maps defined on open subsets of Λ_ϵ. Assume that Λ_ϵ is compact.*

Then for every $\delta > 0$, there exists two family of functions $n_i^* : \mathbb{N}^i \to \mathbb{N}$ and $m_i^* : \mathbb{N}^{2i+1} \times N^{i+1} \to \mathbb{N}$ such that for every pseudo-orbit y_i, $0 \leq i \neq N-1$, of the form

$$y_{i+1} = T^{m_i} \circ S_{j_i} \circ T^{n_i}$$

such that

$$n_i \geq n_i^*(j_0, \ldots, j_i), \quad m_i \geq m^*(n_0, \ldots, n_i, m_0, \ldots, m_{i-1}, j_0, \ldots, j_i),$$

there exists and orbit $z(t)$ of the Hamiltonian flow and $t_0 < \cdots < t_N$ such that

$$d(z(t_i), y_i) < \delta.$$

To prove Proposition 3.3, given any pseudo-orbit of T and S_j, we can use Poincaré recurrence to attach arbitrarily long excursions of iterates of T. This allows us to construct a new pseudo-orbit that satisfies the requirement of Lemma A.1.

In the rest of the section,any unspecified "measure" stands for Lebesgue measure.

Proof of Proposition 3.3. First, rewrite the pseudo-orbit in the following form

$$x_{i+1} = T^{s_i} \circ S_{j_i}(x_i) =: f_i(x_i), \quad 0 \le i \ne N-1$$

where $s_i \ge 0$. Note that each f_i maps positive measure sets to positive measure sets, and zero measure sets to zero measure sets. Since each f_i are continuous, then there exists open balls B_i with $x_i \in B_i \subset B_{\delta/2}(x_i)$ such that

$$f_i(B_i) \subset B_{i+1}.$$

Let n_i^*, m_i^* be the functions given by Lemma A.1 depending on $\delta/2$ and j_i.

By the Poincaré recurrence theorem, there exists $n_0 \ge n_0^*(j_0)$ such that both

$$E_0' = \{x \in B_0 : T^{k_0}(x) \in B_0\},$$

and $T^{n_0}(E_0')$ have positive measure. Since

$$f_0 \circ T^{n_0}(E_0'),$$

has positive measure, there exists $l_0 \ge m_0^*(n_0, j_0) - s_0$ such that

$$F_0' = \{x \in f_0 \circ T^{n_0}(E_0') : T^{l_0}(x) \in f_0 \circ T^{n_0}(E_0')\}$$

has positive measure and so does $F_0 = T^{l_0}(F_0') \subset f_0 \circ T^{n_0}(E_0') \subset B_1$. Set $m_0 = l_0 + s_0$ and $g_0 = T^{l_0} \circ f_0 \circ T^{n_0} = T^{m_0} \circ S_{j_0} \circ T^{n_0}$, and

$$E_0 = g_0^{-1}(F_0) \cap E_0'.$$

Then $g_0(E_0) = T^{l_0}(F_0') = F_0$ and both E_0 and F_0 have positive measure.

Inductively, suppose $E_i, F_i, g_i = T^{m_i} \circ S_{j_i} \circ T^{n_i}$ are constructed such that

$$n_i \ge n_i^*(j_0, \dots, j_i), \quad m_i \ge m_i^*(n_0, \dots, n_i, m_0, \dots, m_{i-1}, j_0, \dots, j_i),$$

satisfies

$$E_i \supset \cdots \supset E_0,$$

$$F_i = g_i \circ \cdots \circ g_0(E_i) \subset B_{i+1}$$

and E_i, F_i have positive measure. See Figure 3 for a diagram of the sets involved in the construction.

We set

$$E_{i+1}' = \{x \in F_i : T^{n_{i+1}}(x) \in F_i\},$$

where $n_{i+1} \ge n_{i+1}^*(j_0, \dots, j_{i+1})$ is chosen such that both E_{i+1}' and $T^{n_{i+1}}(E_{i+1})$ has positive measure. Let $l_{i+1} \ge m_{i+1}^*(n_0, \dots, n_{i+1}, m_0, \dots, m_i, j_0, \dots, j_i) - s_{i+1}$ such that

$$F_{i+1}' = \{x \in f_{i+1} \circ T^{n_{i+1}}(E_{i+1}') : T^{n_{i+1}}(x) \in f_{i+1} \circ T^{n_{i+1}}(E_{i+1}')\}$$

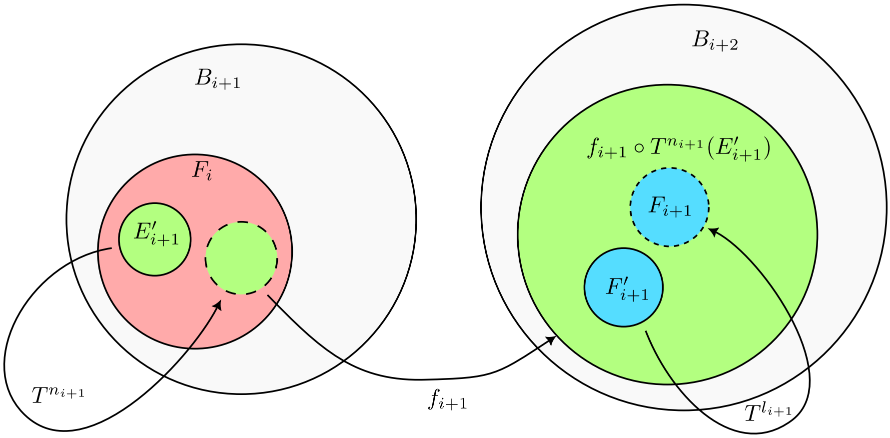

Fig. 3. Diagram for the choice of the sets E_i, F_i

$m_{i+1} = l_{i+1} + s_{i+1}$, $F_{i+1} = T^{l_{i+1}}(F'_{i+1}) \subset f_{i+1} \circ T^{n_{i+1}}(E'_{i+1}) \subset B_{i+2}$. Our construction ensures

$$g_{i+1} \circ \cdots g_0(E_i) = g_{i+1}(F_i) \supset g_{i+1}(E'_{i+1}) \supset F_{i+1},$$

therefore

$$E_{i+1} := (g_{i+1} \circ \cdots g_0(E_i))^{-1} (F_{i+1}) \cap E_i$$

is mapped onto F_{i+1} by $g_{i+1} \circ \cdots g_0(E_i)$.

Apply the construction for $0 \le i \le N-1$, we obtain

$$g_i \circ \cdots g_0(E_{N-1}) \subset g_i \circ \cdots g_0(E_i) = F_i \subset B_{i+1},$$

for all $0 \le i \le N-1$. Pick any $y_0 \in E_N$, then

$$y_{i+1} = g_i(y_i)$$

defines a pseudo-orbit satisfying the condition of Lemma A.1 and

$$d(y_i, x_i) < \delta/2.$$

Apply Lemma A.1, we get a real orbit shadowing y_i within $\delta/2$ distance, which shadows x_i within δ distance. □

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Communicated by C. Liverani