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EXPONENTIALLY SMALL SPLITTING OF SEPARATRICES FOR WHISKERED TORI IN HAMILTONIAN SYSTEMS

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We study the existence of transverse homoclinic orbits in a singular or weakly hyperbolic Hamiltonian with three degrees of freedom as a model for the behavior of a nearly integrable Hamiltonian near a simple resonance. The considered example consists of an integrable Hamiltonian having a two-dimensional hyperbolic invariant torus with fast frequencies $\omega/\sqrt{\varepsilon}$ and coincident whiskers or separatrices, plus a perturbation of order $\mu = \varepsilon^p$ giving rise to an exponentially small splitting of separatrices. We show that asymptotic estimates for the transversality of the intersections can be obtained if ω satisfies certain arithmetic properties. More precisely, we assume that ω is a quadratic vector (i.e., the frequency ratio is a quadratic irrational number) and generalizes good arithmetic properties of the golden vector. We provide a sufficient condition on the quadratic vector ω ensuring that the Poincaré–Melnikov method (used for the golden vector in a previous work) can be applied to establish the existence of transverse homoclinic orbits and, in a more restricted case, their continuation for all values of $\varepsilon \to 0$. Bibliography: 22 titles.

1. Introduction and main results

The detection of *transverse homoclinic orbits* to an invariant object is one of the main tools for proving the existence of chaotic motion in a dynamical system. Such a detection is complicated in the case of a Hamiltonian system ε-close to a completely integrable one. Among KAM tori, there appear generically whiskered tori with noncoincident whiskers, giving rise to the phenomenon called *splitting of separatrices*, which is *exponentially small* with respect to ε.

In the case of a one-dimensional whiskered torus (periodic orbit) of a Hamiltonian with two degrees of freedom, V. F. Lazutkin introduced, in the seminar paper [13], *complex parameterizations* for invariant manifolds, thus associating with the splitting an analytic periodic function with zero mean. The width of the strip of analyticity of this function explicitly appears in the exponent of the splitting, and it turns out that only one (the first) harmonic of the perturbation is relevant for describing the size of the splitting (see [10, 11]).

However, if the dimension of the whiskered tori is greater than one, the expression for the quasi-periodic splitting function is more intricate, because it depends on the arithmetic properties of the frequencies of the whiskered tori. Indeed, the effect of small divisors is present in the most important part of the splitting: in the exponent. This fact was first observed by Chirikov [2], and later on by Lochak [15, 16] and Simó [21], and was first proved by Delshams et al. [7] for the pendulum under a fast quasi-periodic forcing (see also [1]). Later on, the splitting of separatrices for a two-dimensional whiskered torus in a Hamiltonian system with three degrees of freedom was dealt with by Sauzin and co-workers [20, 14], Rudnev and Wiggins [19], Pronin and Treschev [18], and also Simó and Valls [22], who also considered possible homoclinic bifurcations. It is important to say that the main tool that has been used to establish the splitting for whiskered tori with two or more frequencies is the validation of the expression provided by a direct application of the *Poincaré–Melnikov method*.

In fact, in the Hamiltonian setting, it turns out (see [9, 3]) that the splitting vector distance and the Melnikov vector function are the gradients of scalar functions, called the *splitting potential* and the *Melnikov potential*, respectively. This implies that transverse homoclinic orbits to whiskered tori correspond to nondegenerate critical points of the splitting potential.

The arithmetic properties of the frequencies of a whiskered torus are of great importance. As a matter of fact, all currently known rigorous expressions involve only two frequencies and some famous *quadratic* numbers, like the golden number. One substantially uses the theory of continued fractions to separate primary resonances (in the case of the golden number, those associated with the Fibonacci numbers) from other weaker resonances.

In this context, the existence of transverse homoclinic orbits to a two-dimensional whiskered torus, with frequency vector equal to the golden vector, of a Hamiltonian with three degrees of freedom was proved in [20, 14], but not for all values of $\varepsilon \to 0$, because at a certain sequence of values of ε, the dominant harmonics of the splitting function change, and homoclinic bifurcations may take place. Some examples of such bifurcations were described in [22].

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In [6], this result in the same situation was improved with the help of a careful analysis of the Melnikov function and its dominant harmonics and accurate bounds for the size of the error term obtained (by using the flow-box coordinates) in [8]. Indeed, it was shown in [6] that the dominant harmonics of the splitting function correspond to the dominant harmonics in the Melnikov approximation, providing asymptotic estimates for the splitting. With such estimates, one can show the existence of precisely four transverse homoclinic orbits and their *continuation* for all values of the perturbation parameter $\varepsilon \to 0$ (with no bifurcations).

In this paper, we consider some particular perturbations with three degrees of freedom and infinitely many harmonics, and study the question to what extent the results quoted above can be generalized to an arbitrary quadratic frequency vector (i.e., a vector with a quadratic number as the frequency ratio). Using a generalization of the arithmetic properties of the golden vector to other quadratic vectors, we can carry out a suitable analysis of the Melnikov function and its dominant harmonics, as well as of the size of the remaining harmonics. Under a suitable condition on the quadratic vector, we obtain asymptotic estimates for the splitting function, which allow us to establish the existence of a certain number of transverse homoclinic orbits, although some of these orbits may bifurcate for ε close to certain critical values (as in [22]).

In the best case (that of the golden vector and other noble frequency vectors), we can ensure the continuation of transverse homoclinic orbits for all $\varepsilon \to 0$, as in [6]. For some other quadratic vectors, we can at least ensure the existence of transverse homoclinic orbits, although bifurcations of these orbits may occur for some critical values of ε.

Now we give a more precise description of the setting and background and describe the new results obtained in the present work.

1.1. Setup: A singular Hamiltonian with three degrees of freedom.

We consider a Hamiltonian system, with three degrees of freedom, that depends on two perturbation parameters ε and μ. In the canonical coordinates $(x, y, \varphi, I) \in \mathbb{T}\times\mathbb{R}\times\mathbb{T}^2\times\mathbb{R}^2$ with the symplectic form $\mathrm{d}x\wedge\mathrm{d}y+\mathrm{d}\varphi\wedge\mathrm{d}I$, our Hamiltonian is defined by the formulas

$$H(x, y, \varphi, I) = H_0(x, y, I) + \mu H_1(x, \varphi), \tag{1}$$

$$H_0(x, y, I) = \langle\omega_\varepsilon, I\rangle + \frac{1}{2}\langle\Lambda I, I\rangle + \frac{y^2}{2} + \cos x - 1, \tag{2}$$

$$H_1(x, \varphi) = h(x) f(\varphi). \tag{3}$$

We assume without loss of generality that $\varepsilon > 0$ and also $\mu > 0$. The vector of *fast frequencies* ω_ε appearing in (2) will be determined by a *quadratic vector*

$$\omega = (1, \Omega), \qquad \omega_\varepsilon = \frac{\omega}{\sqrt{\varepsilon}}. \tag{4}$$

In other words, the frequency ratio Ω is a quadratic irrational number. Equation (2) contains also a symmetric 2×2 matrix Λ such that H_0 satisfies the *isoenergetic nondegeneracy condition*:

$$\det \begin{pmatrix} \Lambda & \omega \\ \omega^\top & 0 \end{pmatrix} \neq 0. \tag{5}$$

In the perturbation (3), we use the following concrete analytic periodic functions:

$$h(x) = \cos x - \nu \quad \text{with} \quad \nu = 0 \quad \text{or} \quad \nu = 1, \tag{6}$$

$$f(\varphi) = \sum_{k\in\mathcal{Z}} f_k \cos(\langle k, \varphi\rangle - \sigma_k) \quad \text{with} \quad f_k = e^{-\rho|k|} \quad \text{and} \quad \sigma_k \in \mathbb{T}, \tag{7}$$

$$\mathcal{Z} = \{k = (k_1, k_2) \in \mathbb{Z}^2 : k_2 > 0 \quad \text{or} \quad (k_2 = 0, k_1 \geq 0)\} \tag{8}$$

(the set $\mathcal{Z}$ is introduced in order to avoid repetitions in the Fourier expansion of $f(\varphi)$).

The Hamiltonian H_0 (that corresponds to the case $\mu = 0$) has a two-parameter family of two-dimensional whiskered tori given by the equations $I = \text{const}$, $x = y = 0$. The stable and unstable whiskers of each torus coincide, thus forming a unique homoclinic whisker. We will focus our attention on a concrete *whiskered torus*, located at $I = 0$, whose inner flow has ω_ε as the frequency vector. We denote by $\mathcal{W}_0$ the homoclinic whisker associated with this torus, and consider the following parameterization of this whisker:

$$\mathcal{W}_0 : (x_0(s), y_0(s), \theta, 0), \quad s \in \mathbb{R},\ \theta \in \mathbb{T}^n, \tag{9}$$

$$x_0(s) = 4 \arctan e^s, \quad y_0(s) = \frac{2}{\cosh s}. \tag{10}$$

The inner flow on $\mathcal{W}_0$ is given by the equations $\dot{s} = 1$, $\dot{\theta} = \omega\varepsilon$.

The two parameters ε and μ will not be independent. On the contrary, they will be linked by a relation of the form $\mu = \varepsilon^p$ with a suitable $p > 0$ (the smaller p the better), i.e., we consider a *singular* (also called *weakly hyperbolic*, or *a priori stable*) problem for $\varepsilon \to 0$. The main motivation for this singular setting is that it can be regarded as a model for the behavior of a nearly integrable Hamiltonian near a simple resonance (see, for instance, [8, 5]).

Our choice of a quadratic frequency vector in (4) is motivated by the arithmetic properties of such vectors. An important and well-known property is that quadratic vectors satisfy the *Diophantine condition*

$$|\langle k, \omega \rangle| \geq \frac{\gamma}{|k|^\tau} \qquad \forall k \in \mathbb{Z}^2 \setminus \{0\} \tag{11}$$

with $\tau = 1$ and some $\gamma > 0$ (as regards the value of γ, see Remark 1 at the end of Sec. 2). Other important properties of quadratic vectors to be used are discussed in Sec. 2.

Under conditions (5) and (11), the *hyperbolic KAM theorem* implies that for μ small enough, the whiskered torus persists, as well as its local whiskers. We point out that the difference between the two values of ν in (6) is that in the case $\nu = 0$, the whiskered torus persists with some shift and deformation, whereas in the case $\nu = 1$, it remains fixed under the perturbation, though the whiskers do suffer some deformation. The Lyapunov exponent of the torus, which is initially equal to 1, perturbs to a close number b. Futhermore, in the isoenergetic case we consider, the frequency vector ω_ε of the torus perturbs to a close and proportional vector

$$\tilde{\omega}_\varepsilon = b'\omega_\varepsilon = \frac{b'\omega}{\sqrt{\varepsilon}}. \tag{12}$$

The numbers b and b' tend to 1 as $\mu \to 0$, and $b' = 1$ in the case $\nu = 1$ (see [8, Theorem 1] for a precise statement).

As regards the Fourier expansion (7), the constant $\rho > 0$ gives the complex width of analiticity of $f(\varphi)$. In principle, the phases σ_k can be chosen arbitrarily, although a certain quite general condition on these phases will be needed for the validity of our results (see Sec. 4).

1.2. Background: The splitting function and the Poincaré–Melnikov method.

When local whiskers are extended to global ones, one can in general expect the existence of splitting between the stable and unstable whiskers (denoted by $\mathcal{W}^+$ and $\mathcal{W}^-$), since they will no longer coincide. In order to study this splitting, the symplectic *flow-box coordinates* (S, E, φ, I) in some neighborhood containing a piece of both whiskers (and not including the torus, where such coordinates are not valid) were introduced in [8]. In the flow-box coordinates, the Hamiltonian equations take a very simple form: $\dot{S} = b$, $\dot{E} = 0$, $\dot{\varphi} = \tilde{\omega}_\varepsilon$, $\dot{I} = 0$ (recall that b and $\tilde{\omega}_\varepsilon$ are the perturbed Lyapunov exponent and frequencies). Moreover, these coordinates can be constructed in such a way that the stable whisker will be given by a coordinate plane,

$$\mathcal{W}^+ : \quad (s, 0, \theta, 0), \quad |s| \leq s^*,\ \theta \in \mathbb{T}^n, \tag{13}$$

where the parameters (s, θ) are inherited from (9)–(10), with some translation in s. Then the unstable whisker can be parameterized, in the same neighborhood, as follows:

$$\mathcal{W}^- : \quad (s, \mathcal{E}(s, \theta), \theta, \mathcal{M}(s, \theta)), \quad |s| \leq s^*,\ \theta \in \mathbb{T}^n, \tag{14}$$

and the inner flow on both whiskers is given by the equations $\dot{s} = b$, $\dot{\theta} = \tilde{\omega}_\varepsilon$. To study the splitting, it suffices to consider the vector function $\mathcal{M}$, called the *splitting function* (the function $\mathcal{E}$ is directly related to $\mathcal{M}$ by the energy conservation).

The use of the flow-box coordinates implies the quasi-periodicity of the splitting function $\mathcal{M}$, an important property related to its exponential smallness. More precisely, the function $\mathcal{M}$ is $\hat{\omega}_\varepsilon$-*quasiperiodic*:

$$\mathcal{M}(s,\theta) = \mathcal{M}(0,\theta - \hat{\omega}_\varepsilon s), \quad \text{where} \quad \hat{\omega}_\varepsilon := \frac{\tilde{\omega}_\varepsilon}{b} = \frac{b'\omega}{b\sqrt{\varepsilon}}. \tag{15}$$

Another important property of $\mathcal{M}$, related to the Lagrangian properties of the whiskers, is that it is the gradient of a scalar function $\mathcal{L}$, called the *splitting potential* (see also [3]):

$$\mathcal{M}(s,\theta) = \partial_\theta \mathcal{L}(s,\theta)$$

(and hence $\mathcal{M}$ has zero average with respect to θ). Then the transverse homoclinic orbits can be studied on the section $s = 0$ (or any other section $s = \text{const}$) as nondegenerate critical points of $\mathcal{L}(0,\theta)$.

Applying the Poincaré–Melnikov method, one can give a first-order approximation in μ for the splitting in terms of the *Melnikov potential* and the *Melnikov function*, defined in [8] (see also [3]) in terms of an absolutely convergent integral:

$$\begin{aligned} L(s,\theta) &= -\int_{-\infty}^{\infty} [h(x_0(s+bt)) - h(0)] \cdot f(\theta + \tilde{\omega}_\varepsilon t)\mathrm{d}t + \text{const}, \\ M(s,\theta) &= \partial_\theta L(s,\theta). \end{aligned} \tag{16}$$

These functions are also $\hat{\omega}_\varepsilon$-quasiperiodic, because they are defined in terms of the perturbed Lyapunov exponent b and the perturbed frequencies $\tilde{\omega}_\varepsilon$ introduced in (12). As a consequence, the *error term* defined as

$$\mathcal{R}(s,\theta) = \mathcal{M}(s,\theta) - \mu M(s + s^{(0)},\theta) \tag{17}$$

is also $\hat{\omega}_\varepsilon$-quasiperiodic. The number s^0, not very relevant, compensates the translation of the parameterizations (13)–(14) with respect to the initial parameterization (9).

The main difficulty in validating the Poincaré–Melnikov method in the singular case $\mu = \varepsilon^p$ is that the first-order approximation given by the Melnikov function is exponentially small in ε, as shown in Sec. 3. In principle, it turns out that the Poincaré–Melnikov method can be applied only if μ is exponentially small in ε (see, for instance, [3]), but not in our case $\mu = \varepsilon^p$. Nevertheless, exponentially small upper bounds for the error term (17) can be obtained, and the method holds in the singular case if p is large enough. The key point of the proof of such exponentially small estimates is to obtain bounds on *complex domains* of the parameters (s,θ) and use the quasi-periodic properties of the splitting.

Note that the initial homoclinic whisker $\mathcal{W}_0$ can be defined in the complex domain $|\mathrm{Im}s| < \pi/2$, $|\mathrm{Im}\theta| < \rho$. These restrictions are due to the singularity of (10) at $s = \pm i\pi/2$ and to the expression (7) that involves ρ as the width of analyticity. This domain decreases at the successive steps leading to the definition of the splitting function and potential. One of the main achievements of [8] is the construction of flow-box coordinates such that the loss of the complex domain is controlled by a free small parameter δ with $\delta \ll \pi/2$ and $\delta \ll \rho$. Then, choosing $\delta = \varepsilon^a$ for some $a > 0$ and using the fact that the functions involved are analytic and quasi-periodic and have zero average, one can obtain exponentially small estimates (see [8, 6] for more details).

With all these ingredients, estimates for the splitting function $\mathcal{M}(s,\theta)$ in the singular case can be obtained under some restriction $p > p^*$. In the paper [6], where the frequency ω under consideration is the golden vector, we proved the existence of precisely four transverse homoclinic orbits and their continuation for all values of $\varepsilon \to 0$. In fact, a certain improvement of the exponents can be obtained in the case of a fixed torus. Because of this fact, the exponent p^* depends on the value of ν in (6):

$$p^* = 2 \quad \text{if} \quad \nu = 1, \tag{18}$$

$$p^* = 3 \quad \text{if} \quad \nu = 0. \tag{19}$$

1.3. Description of the results.

Our goal is to study the existence of transverse homoclinic orbits for the Hamiltonian (1)–(7), assuming that the frequency vector in (4) is quadratic. Our aim is to study the question to what extent the results obtained in [6] for the golden vector can be generalized to other quadratic frequencies.

Since we deal with the singular case $\mu = \varepsilon^p$, we need to show that in the Poincaré–Melnikov approximation (17) for the whole splitting function $\mathcal{M}(s,\theta)$, the term $\mu M(s + s^{(0)}, \theta)$ (exponentially small in ε) dominates, in some sense, the error term $\mathcal{R}(s,\theta)$. A natural approach is to provide asymptotic estimates (or at least lower bounds) for the dominant harmonics of the Melnikov potential L. As we will show, such dominant harmonics are closely related to the small divisors of the frequency vector ω. At a subsequent step, we will have to check that the estimates obtained for the dominant harmonics of L are large enough to be valid also for the dominant harmonics of the splitting potential $\mathcal{L}$ (recall that $\mathcal{M} = \partial_\theta \mathcal{L}$), showing that they overcome the part originating from $\mathcal{R}$.

Note that the quasi-periodicity (16) of the splitting function $\mathcal{M}(s,\theta)$ allows us to restrict to the section $s = 0$, and the (simple) zeros of $\mathcal{M}(0,\theta)$ give rise to (transverse) homoclinic orbits. These (simple) zeros are given by (nondegenerate) critical points of the splitting potential $\mathcal{L}(0,\theta)$. In our main result (Theorem 6), we give conditions for the existence of simple zeros of $\mathcal{M}(0,\theta)$, with asymptotic estimates of the associated eigenvalues of $\partial_\theta \mathcal{M}$.

Let us give a short summary of the results presented. First, in Sec. 2, we study the *arithmetic properties* of quadratic frequencies, carrying out a complete analysis of the associated *resonances* (strictly speaking, we should call them *quasi-resonances*), which originate from the small divisors appearing in the coefficients of the Melnikov potential. Such an analysis can be carried out due to the *arithmetic properties* of quadratic vectors, and is a direct generalization of the analysis done in [6] for the golden vector. The main idea is that a quadratic vector is always an eigenvector of some unimodular matrix (see [12]). This leads to the classification of such resonances of ω into "primary" and "secondary" ones.

In Sec. 3, we provide estimates for the Fourier coefficients of the splitting potential $\mathcal{L}$, showing what harmonics are *dominant* among those associated with primary resonances, and giving upper bounds for the remaining primary resonances and for all the secondary ones. In order to prove this result, we proceed as in [6], first obtaining estimates for the Fourier coefficients of the Melnikov potential, and then applying the upper bounds for the error term given in [8].

Since we look for nondegenerate critical points of the splitting potential $\mathcal{L}$ on $\mathbb{T}^2$, we need at least the two most dominant harmonics. We show that as ε goes across the critical values ε_n defined in (40), some changes in the dominance occur. In fact, for ε close to ε_n, we have to consider the three most dominant harmonics, because the second and third ones are of the same magnitude.

In Sec. 4, we study the nondegenerate critical points of $\mathcal{L}$ (which correspond to simple zeros of $\mathcal{M}$) and obtain our *main result* (Theorem 6) on the existence of a certain number of *transverse homoclinic orbits*. More precisely, we give an asymptotic estimate for the minimum (in modulus) eigenvalue of the splitting matrix $\partial_\theta \mathcal{M}(0,\theta_*)$ for each zero θ_* of the function $\mathcal{M}(0,\cdot)$. This eigenvalue provides a measure of transversality of homoclinic orbits. In order to prove the continuation of the transverse homoclinic orbits for the example (1)–(7), we assume a quite general condition (59) on the phases of the Fourier expansion of the function $f(\varphi)$ in (7).

In the best case, this result is valid in both the cases of two or three dominant harmonics and ensures the *continuation* (without bifurcations) of the corresponding homoclinic orbits for all values of $\varepsilon \to 0$. Nevertheless, this requires a condition on the quadratic frequency vector $\omega = (1,\Omega)$, which we give in (60) and call the *strong separation condition*, ensuring that the influence of secondary resonances can be neglected in comparison with primary resonances and the required dominant harmonics can always be found among the primary resonances. Unfortunately, it seems that such a condition is satisfied only by the golden vector $\Omega = (\sqrt{5-1}/2)$ and (hence) "noble" vectors (i.e., vectors that can be reduced to the golden vector by a unimodular transformation).

Nevertheless, in other cases one can check the *weak separation condition* (61), which can be used in the case of two dominant harmonics and ensures the existence of transverse homoclinic orbits, at least for ε not very close to the critical values ε_n, but not the continuation of these orbits for all $\varepsilon \to 0$. If this weaker condition is not fulfilled, then the study of transverse homoclinic orbits becomes more involved, because both primary and secondary resonances should be taken into account.

We conclude this introduction by describing some notation used in this paper. We write $|f| \preceq g$ if $|f| \le cg$ for some positive constant c that does not depend on any of the relevant parameters, i.e., ε and μ. Thus we do not describe the (usually complicated) dependence on quantities like $\rho, \Omega, \ldots$ and include this dependence in "constants." We use the notation $f \sim g$ if $g \preceq f \preceq g$. Finally, the notation $f \simeq g$ means simply that the functions f and g are nearly equal, in the sense that their difference can be neglected.

2. Quadratic frequencies

The analysis of small divisors is relatively simple in the case of a *quadratic frequency vector* $\omega = (\omega_1, \omega_2)$, where ω_2/ω_1 is a quadratic irrational number. We assume without loss of generality that ω is of the form

$$\omega = (1, \Omega), \qquad 0 < \Omega < 1,$$

where Ω is a quadratic irrational number. Our aim is to take advantage of the nice properties of quadratic irrationals, generalizing the results given in [6] for the case of the golden number $\Omega = (\sqrt{5} - 1)/2$.

The important feature to be applied is that quadratic vectors are eigenvectors of suitable integer 2×2-matrices. More precisely, a result established in [12] claims that there exists a *unimodular* matrix T (i.e., a square matrix with integer entries and determinant ± 1) having a (unique) eigenvalue λ with $|\lambda| > 1$ whose associated eigenvector is ω. Denoting $\delta = \det T = \pm 1$, the other eigenvalue of T is δ/λ. (In fact, a generalization of the matrix T to higher dimensions is considered in [12]. In our two-dimensional case, the matrix T can be constructed from the continued fraction of the number Ω; see [17] as a related reference).

It is a consequence of Theorem 2 below that for a quadratic vector ω, the small divisors $\langle k, \omega \rangle$ satisfy the Diophantine condition (11) with $\tau = 1$ and some $\gamma > 0$. With this fact in mind, we define, as in [6], for every $k \in \mathbb{Z}^2 \setminus \{0\}$, its associated *"numerator"* as

$$\gamma_k = \gamma_k(\omega) := |\langle k, \omega \rangle| \cdot |k| \tag{20}$$

(for integer vectors, we use the notation $|k| = |k|_1 = |k_1| + |k_2|$), and note that always $\gamma_k \geq \gamma$. We are going to provide a simple classification of the (quasi-)resonances associated with ω according to the size of their numerators γ_k.

We say that a vector $k \in \mathbb{Z}^2 \setminus \{0\}$ is *admissible* (or ω-admissible) if $|\langle k, \omega \rangle| < 1/2$, and denote by $\mathcal{A}$ the set of admissible integer vectors. The analysis of resonances can be restricted to the set $\mathcal{A}$, since for any $k \notin \mathcal{A}$, we have $|\langle k, \omega \rangle| > 1/2$ and hence $\gamma_k > |k|/2$.

We now consider the matrix

$$U = (T^\top)^{-1},$$

whose eigenvalues are $1/\lambda$ and $\delta\lambda$, and we denote their associated eigenvectors by u and v, respectively. It is easy to see that $\langle v, \omega \rangle = 0$.

We emphasize the following fundamental equality:

$$\langle Uk, \omega \rangle = \langle k, U^\top \omega \rangle = \frac{1}{\lambda} \langle k, \omega \rangle. \tag{21}$$

It follows that if $k \in \mathcal{A}$, then also $Uk \in \mathcal{A}$. We say that a vector k is *primitive* if $k \in \mathcal{A}$, but $U^{-1}k \notin \mathcal{A}$. We deduce from (21) that primitive vectors are precisely those satisfying the inequatity

$$\frac{1}{2|\lambda|} < |\langle k, \omega \rangle| < \frac{1}{2}. \tag{22}$$

It is clear that admissible vectors are those of the form $k^0(j) = (-\mathrm{rint}(j\Omega), j)$, where $j \neq 0$ is an integer and $\mathrm{rint}(j\Omega)$ denotes the closest integer to $j\Omega$. Then we have $\langle k^0(j), \omega \rangle = j\Omega - \mathrm{rint}(j\Omega)$. If $k^0(j)$ is primitive, we say that j is also primitive; let us denote by $\mathcal{P}$ the set of primitive integers j.

For any given $j \in \mathcal{P}$, we define the *resonant sequence* generated by j as the following sequence of (admissible) integer vectors:

$$s(j, n) := U^{n-1} k^0(j), \qquad n \geq 1. \tag{23}$$

The following simple result says that such resonant sequences cover the whole set of admissible vectors.

Lemma 1. *For any $k \in \mathcal{A}$, there exist a primitive integer $j \in \mathcal{P}$ and an integer $n \geq 1$, both unique, such that $k = s(j, n)$.*

Proof. If $k \in \mathcal{A}$, then in the sequence $U^{-n}k$, $n \geq 0$, there is a unique primitive vector. Indeed, in view of (21), one has $|\langle U^{-n}k, \omega \rangle| = |\lambda|^n |\langle k\omega \rangle|$, and hence only one of the vectors $U^{-n}k$ satisfies (22). $\square$

The motivation for defining the sequences $s(j, \cdot)$ is that they provide a classification of resonances, because the numerators $\gamma_{s(j,n)}$ become nearly constant as $n \to \infty$. In fact, the numerators $\gamma_{s(j,n)}$ oscillate around a *"limit numerator,"* which we denote by γ_j^*. In the next result we establish the existence of this limit and provide an explicit formula for its computation.

Theorem 2. *For any $j \in \mathcal{P}$, there exists the limit numerator*

$$\gamma_j^* := \lim_{n\to\infty} \gamma_{s(j,n)} = |\langle k^0(j), \omega\rangle| \cdot K(j), \qquad K(j) := \left| k^0(j) - \frac{\langle k^0(j), \omega\rangle}{\langle u, \omega\rangle u} \right|,$$

and the following properties hold:

(a) $\gamma_{s(j,n)} = \gamma_j^* + \mathcal{O}(\lambda^{-2n}), \quad n \geq 1.$

(b) $|s(j,n)| = K(j)|\lambda|^{n-1} + \mathcal{O}(|\lambda|^{-n}), \quad n \geq 1.$

(c) $\dfrac{(1+\Omega)|j| - a}{2|\lambda|} < \gamma_j^* < \dfrac{(1+\Omega)|j| + a}{2}, \quad a = \frac{1}{2}\Big(1 + \frac{|u|}{|\langle u,\omega\rangle|}\Big).$

Proof. We start by writing $k^0(j)$ as the following linear combination of the eigenvectors of U:

$$k^0(j) = c_1 u + c_2 v, \qquad c_1 = \frac{\langle k^0(j), \omega\rangle}{\langle u, \omega\rangle}, \tag{24}$$

where the value of c_1 has been obtained by computing the scalar product of the linear combination with ω. Then we see that

$$|c_2 v| = |k^0(j) - c_1 u| = K(j). \tag{25}$$

We deduce from (24) and the definition (23) that

$$s(j,n) = \frac{c_1}{\lambda^{n-1}} u + c_2 (\delta\lambda)^{n-1} v.$$

Then

$$|s(j,n)| = |\lambda|^{n-1}|c_2 v| + \mathcal{O}(|\lambda|^{-n}),$$
$$|\langle s(j,n), \omega\rangle| = \frac{|\langle c_1 u, \omega\rangle|}{|\lambda|^{n-1}} = \frac{|\langle k^0(j), \omega\rangle|}{|\lambda|^{n-1}},$$

and we obtain the expression

$$\gamma_{s(j,n)} = |\langle k^0(j), \omega\rangle| \cdot |c_2 v| + \mathcal{O}(\lambda^{-2n}),$$

whose limit as $n \to \infty$ is

$$\gamma_j^* = |\langle k^0(j), \omega\rangle| \cdot |c_2 v|.$$

The expressions obtained for $|s(j,n)|$, $\gamma_{s(j,n)}$, and γ_j^*, together with (25), imply most statements of the theorem.

Finally, we have to find upper and lower bounds for the limit γ_j^*. Since j is primitive, we can use (22) to provide bounds for $|\langle k^0(j), \omega\rangle|$. Moreover, we can obtain bounds for $K(j)$ by using the inequalities

$$\begin{aligned} |K(j) - (1+\Omega)|j|| &\leq |K(j) - |k^0(j)|| + ||k^0(j)| - (1+\Omega)|j|| \leq |c_1 u| + \frac{1}{2} \\ &\leq \frac{|u|}{2|\langle u, \omega\rangle|} + \frac{1}{2} = a, \end{aligned}$$

where we have applied (22) to bound $|c_1|$ and have used the formula $|k^0(j)| = |j| + |\mathrm{rint}(j\Omega)|$ along with the fact that $\mathrm{rint}(j\Omega)$ is the closest integer to $j\Omega$. Then the upper and lower bounds for $|\langle k^0(j), \omega\rangle|$ and $K(j)$ imply those for γ_j^*. $\square$

We will always assume without loss of generality that $j > 0$.

The lower bound in (c) shows that, although we cannot expect the limits γ_j^* to be increasing in j, they tend to infinity as $j \to \infty$. This fact means that the main resonances associated with ω can be found among the sequences (23) generated by the first few primitives. We set

$$\gamma^* = \liminf_{|k|\to\infty} \gamma_k = \min_{j\in\mathcal{P}} \gamma_j^* = \gamma_{j_0}^*. \tag{26}$$

Thus the "most resonant" integer vectors are those belonging to the resonant sequence generated by j_0. We call them *primary resonances* and denote them by

$$s_0(n) = s(j_0, n). \tag{27}$$

Now *secondary resonances* are integer vectors belonging to any of the remaining sequences $s(j,\cdot)$, $j \neq j_0$. We also define the "normalized" limit numerators $\tilde{\gamma}_j^*$ in such a way that the minimum of them is $\tilde{\gamma}_{j_0}^* = 1$, and introduce one more parameter $\tilde{\gamma}^{**} \geq 1$ that measures the *separation* between the primary and secondary resonances:

$$\tilde{\gamma}_j^* = \frac{\gamma_j^*}{\gamma^*}, \qquad \tilde{\gamma}^{**} = \min_{j \in \mathcal{P} \setminus \{j_0\}} \tilde{\gamma}_j^*. \tag{28}$$

We point out that the constants γ^* and $\tilde{\gamma}^{**}$ will be of importance for us: γ^* appears in the constant C_0, which is defined in (35) and directly related to exponentially small estimates for the splitting, and $\tilde{\gamma}^{**}$ tells us whether it suffices to consider primary resonances in order to study the splitting and its transversality, or secondary resonances are also significant.

In conclusion, we illustrate the results of this section for several examples of quadratic vectors $\omega = (1, \Omega)$. For each number Ω under consideration, we provide the matrices T and U, the eigenvalue λ (which allows us to decide whether a given integer j is primitive or not), the minimum γ^* of the limit numerators, the separation $\tilde{\gamma}^{**}$, and the first few primitives $k^0(j)$ with associated normalized limits $\tilde{\gamma}_j^*$, as well as a lower bound for the remaining ones. The first example is the golden number studied in [6].

Example 1

$\Omega = (\sqrt{5}-1)/2 = 0.618034$

$T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$U = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$

$\lambda = 1.618034$

$\gamma^* = 0.723607$

$\tilde{\gamma}^{**} = 4$

$k^0(j)$	$\tilde{\gamma}_j^*$
$(-1,1)$	1
$(-2,4)$	4
$(-4,7)$	5
$(-6,9)$	9
$(-7,12)$	11
$(-9,14)$	11
$(-11,17)$	19
$j \geq 18$	> 11.974169

Example 2

$\Omega = \sqrt{2}-1 = 0.414214$

$T = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$

$U = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$

$\lambda = 2.414214$

$\gamma^* = 0.5$

$\tilde{\gamma}^{**} = 2$

$k^0(j)$	$\tilde{\gamma}_j^*$
$(0,1)$	1
$(-1,3)$	2
$(-2,4)$	4
$(-2,6)$	8
$(-3,8)$	7
$(-4,9)$	7
$(-5,11)$	14
$j \geq 12$	> 6.572330

Example 3

$\Omega = (\sqrt{21}-3)/2 = 0.791288$

$T = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}$

$U = \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix}$

$\lambda = 4.791288$

$\gamma^* = 0.390891$

$\tilde{\gamma}^{**} = 3$

$k^0(j)$	$\tilde{\gamma}_j^*$
$(-1,1)$	1
$(-2,2)$	4
$(-2,3)$	5
$(-3,4)$	3
$(-5,6)$	7
$(-6,7)$	15
$(-6,8)$	12
$j \geq 9$	> 4.030927

Example 4

$\Omega = (\sqrt{15}-3)/2 = 0.436492$

$T = \begin{pmatrix} 7 & 2 \\ 3 & 1 \end{pmatrix}$

$U = \begin{pmatrix} 1 & -3 \\ -2 & 7 \end{pmatrix}$

$\lambda = 7.872983$

$\gamma^* = 0.370901$

$\tilde{\gamma}^{**} = 1.5$

$k^0(j)$	$\tilde{\gamma}_j^*$
$(0,1)$	1.5
$(-1,2)$	1
$(-1,3)$	3.5
$(-2,4)$	4
$(-2,5)$	3.5
$(-3,6)$	9
$(-3,8)$	15
$j \geq 9$	> 2.030016

Remarks.

(1) It is an obvious consequence of Theorem 2 that the exponent in the Diophantine condition (11) is $\tau = 1$. As to the constant γ, it can be taken to be the minimum of all the numerators γ_k, $k \neq 0$. Nevertheless, it is more significant to replace γ by the asymptotic value γ^* defined in (26).

(2) We will implicitly assume that the primitive j_0 providing the minimum in (26) is unique, and hence $\tilde{\gamma}^{**} > 1$. In fact, this happens for all cases we have studied, provided that the matrix T is suitably chosen.

(3) We see in the above examples that the limit numerators for different resonant sequences are integer multiples of a certain number. This fact can be proved rigorously (see [6] for the case of the golden number) and implies a wide separation among different limit numerators, except for some of them whose limits may coincide.

3. Dominant harmonics of the splitting potential

To show that the splitting potential $\mathcal{L}$ has nondegenerate critical points, we have to consider at least the two most dominant harmonics in its Fourier expansion. As we will see below, it depends on ε which harmonics are dominant. For some values of ε, we will have to consider the three most dominant harmonics, because the second and third ones can be of the same magnitude.

Taking into account that $\mathcal{L}$ is $\hat{\omega}_\varepsilon$-quasiperiodic, we can consider the following expressions for its Fourier expansion:

$$\mathcal{L}(s,\theta) = \sum_{k\in\mathbb{Z}^2} \mathcal{L}_k^* e^{i\langle k,\theta-\hat{\omega}_\varepsilon s\rangle} = \sum_{k\in\mathcal{Z}} \mathcal{L}_k \cos(\langle k,\theta-\hat{\omega}_\varepsilon s\rangle - \tilde{\sigma}_k), \tag{29}$$

where $\mathcal{L}_k$ and $\tilde{\sigma}_k$ are real and $\mathcal{L}_k \geq 0$ (recall that $\mathcal{Z}$ is defined in (8)). For every $k\in\mathcal{Z}$, the coefficients of the exponential and trigonometric forms are related by the formulas $\mathcal{L}_k^* = \frac{1}{2}\mathcal{L}_k e^{-i\tilde{\sigma}_k}$, $\mathcal{L}_{-k}^* = \overline{\mathcal{L}_k^*} = \frac{1}{2}\mathcal{L}_k e^{i\tilde{\sigma}_k}$.

If the quadratic vector ω satisfies a certain condition, all the involved dominant harmonics of $\mathcal{L}$ will be among the primary resonances: $k = s_0(n)$. In Lemma 4 below, we give an estimate for the dominant harmonics among the primary ones, as well as bounds for both the remaining primary harmonics and all the secondary harmonics.

We recalled in Sec. 1.2 that a first-order approximation in μ for the splitting potential $\mathcal{L}$ is given by the Melnikov potential L defined in (16). Thus we can study the Fourier coefficients of L (in order to find the dominant ones) using an approach similar to that applied in [6] for the golden vector. Applying the bounds on the error term (17) given in [8], we can show that this dominance persists in the whole splitting potential $\mathcal{L}$.

Let us compute the Fourier coefficients of the Melnikov potential (16):

$$\begin{aligned} L(s,\theta) &= -\sum_{k\in\mathcal{Z}\setminus\{0\}} f_k \int_{-\infty}^{\infty} (\cos x_0(s+bt)-1)\cos(\langle k,\theta+\tilde{\omega}_\varepsilon t\rangle - \sigma_k)\mathrm{d}t \\ &= \sum_{k\in\mathcal{Z}\setminus\{0\}} L_k \cos(\langle k,\theta-\hat{\omega}_\varepsilon s\rangle - \sigma_k), \\ L_k &= 2f_k \int_{-\infty}^{\infty} \frac{\cos\langle k,\tilde{\omega}_\varepsilon t\rangle}{\cosh^2 bt}\mathrm{d}t = \frac{2\pi\langle k,\hat{\omega}_\varepsilon\rangle f_k}{b\sinh(\frac{\pi}{2}\langle k,\hat{\omega}_\varepsilon\rangle)} = \frac{2\pi|\langle k,\hat{\omega}_\varepsilon\rangle| e^{-\rho|k|}}{b\sinh|\frac{\pi}{2}\langle k,\hat{\omega}_\varepsilon\rangle|} \end{aligned} \tag{30}$$

(we take $L_0 = 0$ to have zero average). The integral has been computed by residues, and we have also used the formula $\cos x_0(bt) - 1 = -2/\cosh^2 bt$. Note that the value of ν in (6) does not influence the Melnikov potential, and the phases σ_k in the Fourier expansion of $L(0,\theta)$ are the same as in the function $f(\varphi)$ given in (7). According to (17), we can expect $\mathcal{L}_k$ and $\tilde{\sigma}_k$ in (29) to be perturbations of L_k and σ_k.

When analyzing the coefficients, we first proceed in a rough way, in order to motivate the definitions of C_0, ε_n, and $h_i(\varepsilon)$ given below. To estimate the size of the coefficients L_k in (30), we use, as in [6], the arithmetic properties of ω established in Sec. 2. Taking into account the definition of γ_k in (20) and the fact that b and b' are μ-close to 1, we have

$$|\langle k,\hat{\omega}_\varepsilon\rangle| = \left|\langle k, \frac{b'\omega}{b\sqrt{\varepsilon}}\rangle\right| \simeq \frac{\gamma_k}{|k|\sqrt{\varepsilon}}. \tag{31}$$

Then (30) implies the following approximation for the coefficients:

$$L_k \simeq \alpha_k e^{-\beta_k}, \tag{32}$$

where

$$\alpha_k = \frac{4\pi\gamma_k}{|k|\sqrt{\varepsilon}[1-\exp\{-\frac{\pi\gamma_k}{|k|\sqrt{\varepsilon}}\}]}, \quad \beta_k = \rho|k| + \frac{\pi\gamma_k}{2|k|\sqrt{\varepsilon}}. \tag{33}$$

The largest coefficients L_k correspond to the smallest exponents β_k.

A more suitable expression for these exponents is as follows:

$$\beta_k = \frac{C_0\sqrt{\tilde{\gamma}_k}}{2\varepsilon^{1/4}}\Big(\frac{2\rho|k|\varepsilon^{1/4}}{C_0\sqrt{\tilde{\gamma}_k}} + \frac{C_0\sqrt{\tilde{\gamma}_k}}{2\rho|k|\varepsilon^{1/4}}\Big), \tag{34}$$

where we set $\tilde{\gamma}_k = \gamma_k/\gamma^*$ analogously to (28) and introduce the important constant

$$C_0 := \sqrt{2\pi\rho\gamma^*}. \tag{35}$$

From (34) we deduce the lower bound

$$\beta_k \geq \frac{C_0\sqrt{\tilde{\gamma}_k}}{\varepsilon^{1/4}}, \tag{36}$$

which suggests that the size of the exponent β_k is strongly related (if k is admissible) to the sequence $s(j,\cdot)$, defined in (23), to which k belongs, due to the fact that the numerators tend to a constant for each sequence. Indeed, we know from Theorem 2 that for k belonging to a given sequence $s(j,\cdot)$, the limit of the $\tilde{\gamma}_k$'s is the number $\tilde{\gamma}_j^*$ defined in (28). This means that the smallest exponents β_k can be found among the primary resonances $s_0(\cdot)$ defined in (27).

Let us find out what primary resonances give the smallest exponents. Recall the approximations

$$\tilde{\gamma}_{s_0(n)} = 1 + \mathcal{O}(\lambda^{-2n}), \tag{37}$$

$$|s_0(n)| = K(j_0)|\lambda|^{n-1} + \mathcal{O}(|\lambda|^{-n}) \tag{38}$$

given in Theorem 2. Thus, taking $k = s_0(n)$ in (34), we obtain

$$\beta_{s_0(n)} \simeq \frac{C_0}{2\varepsilon^{1/4}}\Big(\frac{2\rho K(j_0)|\lambda|^{n-1}\varepsilon^{1/4}}{C_0} + \frac{C_0}{2\rho K(j_0)|\lambda|^{n-1}\varepsilon^{1/4}}\Big) = \frac{C_0 g_n(\varepsilon)}{\varepsilon^{1/4}}, \tag{39}$$

where we have introduced the decreasing sequences

$$\varepsilon_n := \Big(\frac{C_0}{2\rho K(j_0)|\lambda|^{n-1}}\Big)^4 = \frac{\varepsilon_0}{\lambda^{4n}}, \quad \varepsilon'_n := \sqrt{\varepsilon_n\varepsilon_{n-1}} = \frac{\varepsilon_0}{\lambda^{4n-2}} \tag{40}$$

and the functions

$$g_n(\varepsilon) := \frac{1}{2}\Big[\Big(\frac{\varepsilon}{\varepsilon_n}\Big)^{1/4} + \Big(\frac{\varepsilon_n}{\varepsilon}\Big)^{1/4}\Big] = \cosh\Big(\frac{\log\varepsilon - \log\varepsilon_n}{4}\Big) = g_0(\lambda^{4n}\varepsilon),$$

which contain the main information on the size of $\beta_{s_0(n)}$. It is clear that each g_n has its minimum at $\varepsilon = \varepsilon_n$. Note that the graph of g_n, as a function of $\log\varepsilon$, is simply the graph of g_0 translated by $4n\log|\lambda|$. This is illustrated in Fig. 1, where we use the logarithmic scale for ε for the sake of clarity. Note that for large n, the neglected terms in (37)–(38) become smaller, and the approximation obtained from the function $g_n(\varepsilon)$ becomes better.

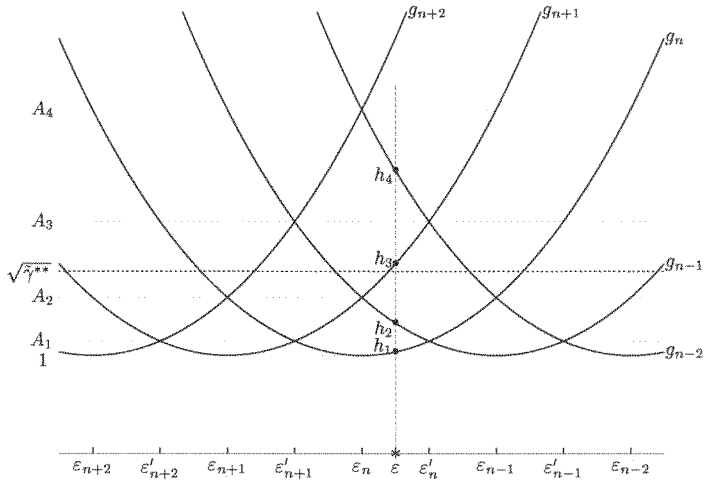

FIG. 1. The functions $h_i(\varepsilon)$ (using the logarithmic scale for ε).

We now consider the intervals $\mathcal{I}_n = [\varepsilon'_{n+1}, \varepsilon'_n]$, $\mathcal{I}'_n = [\varepsilon'_{n+1}, \varepsilon_n]$, and $\mathcal{I}''_n = [\varepsilon_n, \varepsilon'_n]$ for $n \geq 1$, and define the following functions:

$$\begin{aligned} &h_1(\varepsilon) = g_n(\varepsilon) && \text{for } \varepsilon \in \mathcal{I}_n, \\ &h_2(\varepsilon) = g_{n+1}(\varepsilon),\ h_3(\varepsilon) = g_{n-1}(\varepsilon),\ h_4(\varepsilon) = g_{n+2}(\varepsilon) && \text{for } \varepsilon \in \mathcal{I}'_n, \\ &h_2(\varepsilon) = g_{n-1}(\varepsilon),\ h_3(\varepsilon) = g_{n+1}(\varepsilon),\ h_4(\varepsilon) = g_{n-2}(\varepsilon) && \text{for } \varepsilon \in \mathcal{I}''_n. \end{aligned} \tag{41}$$

Connecting the successive intervals $\mathcal{I}_n$, we see that these functions are continuous for all ε from the interval $0 < \varepsilon < \varepsilon'_1$ and satisfy the equation

$$h_i(\lambda^4 \varepsilon) = h_i(\varepsilon) \tag{42}$$

for any ε. In other words, the functions h_i are $4\log|\lambda|$-periodic in $\log\varepsilon$. See Fig. 1 for an illustration of the functions $h_i(\varepsilon)$.

An equivalent way to introduce these functions is to set

$$h_1(\varepsilon) = \frac{1}{2}\left[\left(\frac{\varepsilon}{\varepsilon_1}\right)^{1/4} + \left(\frac{\varepsilon_1}{\varepsilon}\right)^{1/4}\right] \quad \text{if} \quad \varepsilon \in \mathcal{I}_1,$$

$$h_2(\varepsilon) = \begin{cases} \frac{1}{2}\left[\left(\frac{\varepsilon}{\varepsilon_2}\right)^{1/4} + \left(\frac{\varepsilon_2}{\varepsilon}\right)^{1/4}\right] & \text{if} \quad \varepsilon \in \mathcal{I}'_1, \\ \frac{1}{2}\left[\left(\frac{\varepsilon}{\varepsilon_0}\right)^{1/4} + \left(\frac{\varepsilon_0}{\varepsilon}\right)^{1/4}\right] & \text{if} \quad \varepsilon \in \mathcal{I}''_1, \end{cases}$$

and similarly for $h_3(\varepsilon)$ and $h_4(\varepsilon)$, and extend them according to (42).

Defining the constants

$$A_i = \frac{1}{2}(|\lambda|^{i/2} + |\lambda|^{-i/2}), \tag{43}$$

we can easily check the following bounds for the functions $h_i(\varepsilon)$:

$$1 \le h_1(\varepsilon) \le A_1 \le h_2(\varepsilon) \le A_2 \le h_3(\varepsilon) \le A_3 \le h_4(\varepsilon) \le A_4,$$

where equalities can take place only for $\varepsilon = \varepsilon_n, \varepsilon'_n$. More precisely, for $\varepsilon = \varepsilon_n$ we have $h_1 < h_2 = h_3 < h_4$, and for $\varepsilon = \varepsilon'_n$ we have $h_1 = h_2 < h_3 = h_4$ (see again Fig. 1).

For any given $\varepsilon < \varepsilon_1$, we define $N_i = N_i(\varepsilon)$, $i = 1, 2, 3, 4$, as the four integers $n \ge 1$ minimizing $g_n(\varepsilon)$. This means that

$$g_{N_1}(\varepsilon) \le g_{N_2}(\varepsilon) \le g_{N_3}(\varepsilon) \le g_{N_4}(\varepsilon) \le g_n(\varepsilon) \quad \forall n \ne N_1, N_2, N_3, N_4. \tag{44}$$

For ε belonging to a particular interval $\mathcal{I}_n$, the first minimum is given by $N_1 = n$. The second, third, and fourth minima are $N_2 = n \pm 1$, $N_3 = n \mp 1$, and $N_4 = n \pm 2$, respectively, and the signs depend on the subinterval to which ε belongs: $\mathcal{I}'_n$ or $\mathcal{I}''_n$. Thus the integers N_i are *consecutive* (but not ordered). The main fact to be used is that the values of the four minima are given by the functions h_i defined in (41). Indeed, one easily checks that

$$g_{N_i}(\varepsilon) = h_i(\varepsilon), \quad i = 1, 2, 3, 4. \tag{45}$$

Note that there is some ambiguity in the definition of $N_i(\varepsilon)$ at the endpoints of the intervals, but the important fact is that these endpoints are critical values at which some of the numbers $N_i(\varepsilon)$ giving the minima change when ε goes across them.

For brevity, we also denote by

$$S_i = S_i(\varepsilon) := s_0(N_i(\varepsilon)), \quad i = 1, 2, 3, 4,$$

the primary resonances indexed by the minimizing integers. As a consequence of Theorem 2 and the definition of N_i, we easily obtain the following estimate, to be used later:

$$|S_i| \sim |\lambda|^{N_i} \sim \varepsilon^{-1/4}, \quad i = 1, 2, 3, 4. \tag{46}$$

The next lemma implies that the three most dominant harmonics of the splitting potential among the primary ones are the (consecutive) harmonics corresponding to S_1, S_2, and S_3, giving an asymptotic estimate for the coefficients $\mathcal{L}_{S_i}$, as well as a bound comparing the phases $\tilde{\sigma}_k$ with the original phases σ_k in (7) (the difference is caused by the translation $s^{(0)}$ appearing in (17)). In addition, the lemma provides an estimate for the sum of all the coefficients $\mathcal{L}_k$ (recall that $\mathcal{L}_k \ge 0$) associated with primary resonances except the l dominant ones ($0 \le l \le 3$) in terms of the first neglected harmonic among the primary ones, $\mathcal{L}_{S_{l+1}}$, as well as an upper bound for the sum of all the coefficients $\mathcal{L}_k$ associated with secondary resonances. The sum of the two bounds can be regarded as a bound on the difference between the splitting potential and its main part given by the dominant harmonics. In fact, since we are interested in some derivative of the Melnikov potential, we consider the sum of (positive) quantities of type $|k|^m \mathcal{L}_k$. The constant C_0 in the exponentials was defined in (35).

We recall that the notation "$\preceq$" and "$\sim$" was introduced at the end of Sec. 1.3.

Lemma 3. *Assume that* $\varepsilon \preceq 1$ *and* $\mu = \varepsilon^p$, $p > p^*$, *with* p^* *as defined in* (18)–(19). *Then*

(a) $\mathcal{L}_{S_i} \sim \frac{\mu}{\varepsilon^{1/4}} \exp\left\{ -\frac{C_0 h_i(\varepsilon)}{\varepsilon^{1/4}} \right\}$,
$|\tilde{\sigma}_{S_i} - \sigma_{S_i} - s^{(0)}\langle S_i, \hat{\omega}_\varepsilon \rangle| \preceq \frac{\mu}{\varepsilon^{p^*}}, \quad i = 1,2,3,4.$

(b) $\sum_{\substack{k \in s_0(\cdot) \\ k \neq S_1,\dots,S_l}} |k|^m \mathcal{L}_k \sim \frac{1}{\varepsilon^{m/4}} \mathcal{L}_{S_{l+1}}, \quad 0 \le l \le 3,\ m \ge 0.$

(c) $\sum_{k \notin s_0(\cdot)} |k|^m \mathcal{L}_k \preceq \frac{\mu}{\varepsilon^{(m+1)/4}} \exp\left\{ -\frac{C_0 \sqrt{\tilde{\gamma}^{**}}}{\varepsilon^{1/4}} \right\}, \quad m \ge 0.$

Proof. The proof proceeds essentially as in [6], and here we give only a sketch of the proof. The main idea is to obtain the results for the coefficients $\mathcal{L}_k$ of the splitting potential by comparing them with the coefficients L_k of the Melnikov potential and using the bound for the error term (17) given in [8]. In the notation (29), the Fourier coefficients (in the exponential form) are $\mathcal{M}_k^* = ik\mathcal{L}_k^*$ and $M_k^* = ikL_k^*$, respectively. Thus we see from (17) that the Fourier coefficients of the error term $\mathcal{R}(s,\theta)$ are $\mathcal{R}_k^* = ik(\mathcal{L}_k^* - \mu L_k^* e^{-is^{(0)}\langle k\hat{\omega}_\varepsilon\rangle})$, $k \neq 0$, and taking modulus and argument yields

$$|\mathcal{L}_k - \mu L_k| \preceq \frac{|\mathcal{R}_k^*|}{|k|}, \qquad |\tilde{\sigma}_k - \sigma_k - s^{(0)}\langle k\hat{\omega}_\varepsilon\rangle| \preceq \frac{|\mathcal{R}_k^*|}{|k|\mu L_k}. \tag{47}$$

In [8, Theorem 10] (see also [6, Theorem 0]), there is a bound for the error term on the complex domain $|\mathrm{Im} s| \le \pi/2 - \delta$, $|\mathrm{Im}\rho| \le \rho| - \delta$, where $\delta > 0$ is a small reduction. With $\delta = \varepsilon^{1/4}$, the bound on such a domain can be written as

$$|\mathcal{R}| \preceq \frac{\mu^2}{\varepsilon^q},$$

with $q = 5/2$ if $\nu = 1$, and $q = 7/2$ if $\nu = 0$ (recall that there is some improvement in the case of a fixed torus). Since the function $\mathcal{R}$ is $\hat{\omega}_\varepsilon$-quasi-periodic, applying a standard result (see, for instance, [8, Lemma 11]) yields the following bound for its Fourier coefficients:

$$\begin{gathered} |\mathcal{R}_k^*| \preceq \frac{\mu^2}{\varepsilon^q} e^{-\tilde{\beta}_k} \simeq \frac{\mu^2}{\varepsilon^q} e^{-\beta_k}, \\ \tilde{\beta}_k = (\rho - \varepsilon^{1/4})|k| + (\frac{\pi}{2} - \varepsilon^{1/4})|\langle k, \hat{\omega}_\varepsilon\rangle| \simeq \beta_k, \end{gathered} \tag{48}$$

where, as in [6], the perturbation terms with $\varepsilon^{1/4}$ in the exponent $\tilde{\beta}_k$ can be neglected due to the denominator $\varepsilon^{1/4}$ in (34), and the μ-small terms in $\hat{\omega}_\varepsilon$ can be neglected as in (31).

To establish (a) and (b), we need to consider only primary resonances. For the coefficients $\mathcal{L}_{s_0(n)}$ of the splitting potential, at the first step we consider the approximation given by the coefficients $L_{s_0(n)}$ of the Melnikov potential and look for the dominant ones. Then, at the second step, we show that if $\mu = \varepsilon^p$ with $p > p^*$, then this dominance remains unchanged when adding the error term (17).

Thus we first look for the largest coefficients $L_{s_0(n)}$, i.e., the smallest exponents $\beta_{s_0(n)}$. We see from (39) and (44) that the four smallest exponents are those obtained for $n = N_i$. Since the functions $h_i(\varepsilon)$ have been defined in such a way that (45) holds, we deduce that

$$\beta_{S_i} \simeq \frac{C_0 h_i(\varepsilon)}{\varepsilon^{1/4}}, \qquad i = 1,2,3,4.$$

We see from (46) that $\alpha_{S_i} \sim \varepsilon^{-1/4}$ in (32)–(33), and hence

$$L_{S_i} \sim \frac{1}{\varepsilon^{1/4}} \exp\left\{ -\frac{C_0 h_i(\varepsilon)}{\varepsilon^{1/4}} \right\}, \qquad i = 1,2,3,4. \tag{49}$$

Recall that we are dealing with approximations, and we actually have a perturbation of the situations described, due to the terms neglected in (31) and (37)–(38). As thoroughly explained in [6], if we take into account the size of the terms neglected, we can see that for our choice of μ, the asymptotic estimates for the dominant coefficients remain the same.

Now, to estimate the size of the coefficients $\mathcal{L}_{S_i}$ of the splitting potential, we use (47)–(48) together with (46) and obtain

$$|\mathcal{L}_{S_i} - \mu L_{S_i}| \preceq \frac{\mu^2}{\varepsilon^{q-1/4}} \exp\Big\{ -\frac{C_0 h_i(\varepsilon)}{\varepsilon^{1/4}} \Big\}, \qquad i = 1, 2, 3, 4.$$

This upper bound is dominated by the term $|\mu L_{S_i}|$, estimated in (49), provided that $\mu \preceq \varepsilon^p$ with $p > q-1/2 = p^*$. Taking p^* as defined in (18)–(19), we obtain the first assertion of (a), and the second one can be proved in a similar way.

The proof of (b) works as in [6]: we bound the sum of the coefficients $|s_0(n)|^m \mathcal{L}_{s_0(n)}$, excluding some (consecutive) dominant ones ($n \neq N_1, \dots, N_l$), by a geometric series whose main term is the next dominant harmonic ($n = N_{l+1}$). It can also be shown that the term $|s_0(n)|^m$ does not affect this dominance.

Finally, we can prove (c) in a similar way, bounding the sum of the secondary coefficients $|k|^m \mathcal{L}_k$ with $k \in s(j,\cdot)$, $j \neq j_0$, by a geometric series. Now an upper bound for the main term of this series can be obtained from the lower bound (36) for the exponent β_k by using the fact that the normalized numerators $\tilde{\gamma}_k$, for $k \in s(j,\cdot)$, tend to $\tilde{\gamma}_j^* \geq \tilde{\gamma}^{**}$. In fact, the sum also includes the coefficients associated with nonadmissible k, i.e., with k that do not belong to any sequence $s(j,\cdot)$ (see Sec. 2). Such coefficients are clearly dominated by admissible ones, since we have $\gamma_k > |k|/2$ and we see from (33) that $\beta_k \succeq 1/\sqrt{\varepsilon}$ for the nonadmissible case. $\square$

Remarks.

(1) We have shown that the harmonics S_i are the most dominant among the primary ones, but it is not excluded that some secondary harmonic can be more dominant than some of the S_i's. This depends on the relation between the separation $\tilde{\gamma}^{**}$ and the constants A_i introduced in (43). Thus if $A_i \leq \sqrt{\tilde{\gamma}^{**}} < A_{i+1}$, we can ensure that the i most dominant harmonics are primary (see again Fig. 1).

(2) To give a more refined bound in (c), we could define some functions (periodic in $\log \varepsilon$) for the secondary resonances, analogous to the functions $h_i(\varepsilon)$ introduced in (41). Then the number of (primary or secondary) dominant harmonics for which asymptotic estimates can be given could be larger than in the previous remark. It is not hard to carry out this approach for concrete examples of quadratic frequencies, but it seems more involved to give its general description.

(3) Assume that we consider a perturbation $f(\varphi)$ having only primary harmonics, instead of the "full" series considered in (7). Then the Melnikov potential L has only primary harmonics. The splitting potential $\mathcal{L}$ can be "full," but its secondary harmonics are μ^2-small (since they originate from the error term). Then all the dominant harmonics are among the primary ones, and this is not prevented by the separation $\tilde{\gamma}^{**}$. An example of this type is given in [4], where the frequency vector is the golden vector and the perturbation has only the harmonics associated with the Fibonacci numbers.

4. Critical points of the splitting potential

In this section, we are going to use the estimates given in Lemma 3 to show that the splitting potential $\mathcal{L}(0,\theta)$ has nondegenerate critical points (fixing $s = 0$). First we study critical points for the approximations given by the two or three most dominant harmonics among the primary ones:

$$\mathcal{L}^{(2)}(\theta) = \sum_{i=1,2} \mathcal{L}_{S_i} \cos(\langle S_i, \theta\rangle - \tilde{\sigma}_{S_i}),$$
$$\mathcal{L}^{(3)}(\theta) = \sum_{i=1,2,3} \mathcal{L}_{S_i} \cos(\langle S_i, \theta\rangle - \tilde{\sigma}_{S_i}). \tag{50}$$

Afterwards, we discuss the persistence of these critical points in the whole function $\mathcal{L}(0,\theta)$. As the functions $h_i(\varepsilon)$ defined in (41) suggest, it seems natural to consider the two dominant harmonics for most values of ε, and three dominant harmonics for ε close to a critical value ε_n. Nevertheless, we emphasized in Remark 1 after Lemma 3 that some of the dominant harmonics may be secondary, depending on the separation $\tilde{\gamma}^{**}$. Then the approximations defined in (50) would not be good enough.

Anyway, we begin by studying the function $\mathcal{L}^{(2)}$ for $\varepsilon \neq \varepsilon_n$ and the function $\mathcal{L}^{(3)}$ for $\varepsilon \neq \varepsilon'_n$. To fix ideas, we look at concrete intervals: we assume that $\varepsilon \in (\varepsilon_n, \varepsilon_{n-1})$ in the first case, and $\varepsilon \in (\varepsilon'_{n+1}, \varepsilon'_n)$ in the second case. Recalling Fig. 1, note that

$$\begin{aligned}
&S_1 = s_0(n),\ S_2 = s_0(n+1),\ S_3 = s_0(n-1) && \text{for } \varepsilon \in (\varepsilon'_{n+1}, \varepsilon_7 n),\\
&S_1 = s_0(n),\ S_2 = s_0(n-1),\ S_3 = s_0(n+1) && \text{for } \varepsilon \in (\varepsilon_n, \varepsilon'_n),\\
&S_1 = s_0(n-1),\ S_2 = s_0(n) && \text{for } \varepsilon \in (\varepsilon'_n, \varepsilon_{n-1}).
\end{aligned}$$

In order to obtain a simpler expression for the functions $\mathcal{L}^{(i)}(\theta)$, in both cases we make the linear change $(\theta_1, \theta_2) \mapsto (\psi_1, \psi_2)$ defined by the formulas

$$\psi_1 = \langle s_0(n-1), \theta \rangle - \tilde{\sigma}_{s_0(n-1)}, \quad \psi_2 = \langle s_0(n), \theta \rangle - \tilde{\sigma}_{s_0(n)}, \tag{51}$$

which can be written as

$$\psi = \mathcal{A}_n \theta - b_n, \quad \text{where} \quad \mathcal{A}_n = \begin{pmatrix} s_0(n-1)^\top \\ s_0(n)^\top \end{pmatrix}, \quad b_n = \begin{pmatrix} \tilde{\sigma}_{s_0(n-1)} \\ \tilde{\sigma}_{s_0(n)} \end{pmatrix}. \tag{52}$$

This change is not always one-to-one on $\mathbb{T}^2$. Indeed, setting

$$\delta = \det U = \pm 1, \qquad \tau = \operatorname{tr} U,$$

we have $U^2 = \tau U - \delta \mathrm{Id}$, and we obtain the following recurrence relation for the primary resonances:

$$s_0(n+1) = \tau s_0(n) - \delta s_0(n-1).$$

By induction, we deduce from this relation that $|\det \mathcal{A}_n| = \kappa$ for all n, where

$$\kappa := |\det \mathcal{A}_2| = |\det(k^0(j_0), U k^0(j_0))| \tag{53}$$

(this is a nonvanishing integer, because $k^0(j_0)$ is not an eigenvector of U). This means that the change (52) takes κ points (θ_1, θ_2) to one point (ψ_1, ψ_2). Under the change (52), the functions $\mathcal{L}^{(2)}(\theta)$ and $\mathcal{L}^{(3)}(\theta)$ go, respectively, to the following ones:

$$\begin{aligned} \mathcal{K}^{(2)}(\psi) &= A \cos \psi_1 + B \cos \psi_2, \\ \mathcal{K}^{(3)}(\psi) &= B\eta(1-Q) \cos \psi_1 + B \cos \psi_2 + B\eta Q \cos(\tau \psi_2 - \delta \psi_1 - \triangle\tilde{\sigma}), \end{aligned} \tag{54}$$

where

$$\begin{aligned} &A = \mathcal{L}_{s_0(n-1)}, \quad B = \mathcal{L}_{s_0(n)}, \\ &\eta = \frac{\mathcal{L}_{s_0(n-1)} + \mathcal{L}_{s_0(n+1)}}{\mathcal{L}_{s_0(n)}}, \qquad Q = \frac{\mathcal{L}_{s_0(n+1)}}{\mathcal{L}_{s_0(n-1)} + \mathcal{L}_{s_0(n+1)}}, \\ &\triangle\tilde{\sigma} = \tilde{\sigma}_{s_0(n+1)} - \tau \tilde{\sigma}_{s_0(n)} + \delta \tilde{\sigma}_{s_0(n-1)} \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}. \end{aligned} \tag{55}$$

Note that A, B, η, and Q are positive, because so are the coefficients $\mathcal{L}_k$ in (29). Looking at $\mathcal{K}^{(2)}$, we have $B = \mathcal{L}_{S_1}$, $A = \mathcal{L}_{S_2}$ for $\varepsilon \in (\varepsilon_n, \varepsilon'_n)$, and $A = \mathcal{L}_{S_1}$, $B = \mathcal{L}_{S_2}$ for $\varepsilon \in (\varepsilon'_n, \varepsilon_{n-1})$, i.e., the first and second dominant harmonics swap when ε goes across the value ε'_n.

On the contrary, when looking at $\mathcal{K}^{(3)}$, we have $B = \mathcal{L}_{S_1}$ for any $\varepsilon \in (\varepsilon'_{n+1}, \varepsilon'_n)$. Thus the first dominant harmonic of $\mathcal{K}^{(3)}$ is always $\cos \psi_2$, whereas the second and third ones swap when ε goes across ε_n. Note that η measures the size of the second and third harmonics with respect to the first one, and Q is an indicator of the relative weight of the second and third harmonics ($0 < Q < 1$). We study $\mathcal{K}^{(3)}$ in terms of η and Q, considering η as a perturbation parameter (note that $\eta \sim \mathcal{L}_{S_2}/\mathcal{L}_{S_1}$ is small except for ε close to the endpoints ε'_n and ε'_{n+1}). However, we have to point out that η and Q are not independent parameters, because they are both linked to ε.

In the next lemma, we show the existence of four critical points for η small enough and any Q, provided that the difference of phases $\triangle\tilde{\sigma} \in \mathbb{T}$ is not very close to 0 or π (mod 2π). To measure this closeness, we set

$$\tilde{\sigma}^* = \min(|\triangle\tilde{\sigma}|, |\triangle\tilde{\sigma} - \pi|).$$

Lemma 4.

(a) *The function* $\mathcal{K}^{(2)}$ *has exactly four critical points, all nondegenerate:* $\psi^{(2)}_{(1)} = (0,0)$, $\psi^{(2)}_{(2)} = (0,\pi)$, $\psi^{(2)}_{(3)} = (\pi,0)$, *and* $\psi^{(2)}_{(4)} = (\pi,\pi)$. *At the critical points,* $|\det \mathrm{D}^2\mathcal{K}^{(2)}(\psi^{(2)}_{(j)})| = AB$.

(b) *Assume that* $\tilde{\sigma}^* > 0$ *and define* $E^{(\pm)}$ *and* $\alpha^{(\pm)}$ *by the formulas*

$$E^{(\pm)} = \sqrt{1 - 2Q(1-Q)(1-(\pm 1)^\tau \cos\triangle\tilde{\sigma})},$$
$$\cos\alpha^{(\pm)} = \frac{(1-Q) + (\pm 1)^\tau Q\cos\triangle\tilde{\sigma}}{E^{(\pm)}}, \quad \sin\alpha^{(\pm)} = -\frac{(\pm 1)^\tau \delta Q \sin\triangle\tilde{\sigma}}{E^{(\pm)}}.$$

Then for any $Q \in [0,1]$ *and* $0 < \eta \preceq \tilde{\sigma}^*$, *the function* $\mathcal{K}^{(3)}$ *has exactly four critical points, all nondegenerate:* $\psi^{(3)}_{(j)} = \psi^{(3)}_{(j),0} + \mathcal{O}(\eta)$, $j = 1,2,3,4$, *where* $\psi^{(3)}_{(1),0} = (\alpha^{(+)}, 0)$, $\psi^{(3)}_{(2),0} = (\alpha^{(-)}, \pi)$, $\psi^{(3)}_{(3),0} = (\alpha^{(-)} + \pi, 0)$, *and* $\psi^{(3)}_{(4),0} = (\alpha^{(-)} + \pi, \pi)$. *At the critical points,*

$$|\det \mathrm{D}^2\mathcal{K}^{(3)}(\psi^{(3)}_{(1,3)})| = B^2(E^{(+)}\eta + \mathcal{O}(\eta^2)),$$
$$|\det \mathrm{D}^2\mathcal{K}^{(3)}(\psi^{(3)}_{(2,4)})| = B^2(E^{(-)}\eta + \mathcal{O}(\eta^2)).$$

Proof. We do not prove (a) because it is very simple. On the contrary, the proof of (b) requires more efforts, and it proceeds as in [6]. The critical points of $\mathcal{K}^{(3)}$ are the solutions of the following system of equations:

$$\sin\psi_2 = -\eta\delta\tau(1-Q)\sin\psi_1, \quad (1-Q)\sin\psi_1 - \delta Q\sin(\tau\psi_2 - \delta\psi_1 - \triangle\tilde{\sigma}) = 0. \tag{56}$$

It is clear that for η small enough, the solutions of the first equation of (56) are two curves in $\mathbb{T}^2$. One of these curves is η-close to the line $\psi_2 = 0$, and the other one is η-close to the line $\psi_2 = \pi$. To obtain the solutions of (56) on the first curve, we substitute $\psi_2 = \mathcal{O}(\eta)$ into the second equation and obtain the equation $F^{(+)}_\eta(\psi_1) = 0$ with

$$\begin{aligned} F^{(+)}_\eta(\psi_1) &= (1-Q)\sin\psi_1 + Q\sin(\psi_1 + \delta\triangle\tilde{\sigma}) + \mathcal{O}(\eta) \\ &= E^{(+)}\sin(\psi_1 - \alpha^{(+)}) + \mathcal{O}(\eta). \end{aligned}$$

For $\eta = 0$, the solutions are clearly $\alpha^{(+)}$ and $\alpha^{(+)} + \pi$, except for the case $E^{(+)} = 0$ (prohibited by the condition $\tilde{\sigma}^* > 0$). Note that $E^{(+)} \geq \sqrt{(1+\cos\triangle\tilde{\sigma})/2} \succeq \tilde{\sigma}^*$ and, consequently, these solutions persist for $\eta \preceq \tilde{\sigma}^*$. The obtained perturbed solutions give rise to the critical points $\psi^{(3)}_{(1)}$ and $\psi^{(3)}_{(3)}$.

Analogously, we can substitute $\psi_2 = \pi + \mathcal{O}(\eta)$ into the second equation of (56), obtaining the equation $F^{(-)}_\eta(\psi_1) = 0$ with

$$\begin{aligned} F^{(-)}_\eta(\psi_1) &= (1-Q)\sin\psi_1 + (-1)^\tau Q\sin(\psi_1 + \delta\triangle\tilde{\sigma}) + \mathcal{O}(\eta) \\ &= E^{(-)}\sin(\psi_1 - \alpha^{(-)}) + \mathcal{O}(\eta), \end{aligned}$$

whose solutions are now η-perturbations of $\alpha^{(-)}$ and $\alpha^{(-)} + \pi$, except for the case $E^{(-)} = 0$ (also prohibited), and they lead to the critical points $\psi^{(3)}_{(2)}$ and $\psi^{(3)}_{(4)}$.

The determinant can be easily computed. We have

$$\det \mathrm{D}^2\mathcal{K}^{(3)}(\psi) = B^2(\eta\cos\psi_2 \cdot ((1-Q)\cos\psi_1 + Q\cos(\tau\psi_2 - \delta\psi_1 - \triangle\tilde{\sigma})) + \mathcal{O}(\eta^2))$$

for any $\psi \in \mathbb{T}^2$. At the point $\psi^{(3)}_{(1)} = (\alpha^{(+)}, 0) + \mathcal{O}(\eta)$, we obtain

$$\det \mathrm{D}^2\mathcal{K}^{(3)}(\psi^{(3)}_{(1)}) = B^2(\eta(F^{(+)}_0)'(\alpha^{(+)}) + \mathcal{O}(\eta^2)) = B^2(E^{(+)}\eta + \mathcal{O}(\eta^2)),$$

and similarly with $\psi^{(3)}_{(2)}$, $\psi^{(3)}_{(3)}$, and $\psi^{(3)}_{(4)}$. $\square$

Remarks.

(1) The quantity $\tilde{\sigma}^*$, which measures the distance from $\triangle\tilde{\sigma}$ to the "forbidden" values 0 and π (mod 2π), depends on n, as we have seen in (55). However, in Theorem 6 we will assume that $\tilde{\sigma}^*$ is greater than a concrete positive constant (independent of n) by imposing a simple condition on the phases $\sigma_{s_0(n)}$ of the initial perturbation (7).

(2) We can give a description of the *continuation* of the critical points of $\mathcal{K}^{(3)}$ as Q goes from 0 to 1 (recall that this corresponds to transferring the second dominance from the harmonic $\cos\psi_1$ to the harmonic $\cos(\tau\psi_2 - \delta\psi_1 - \triangle\tilde{\sigma})$). Assuming, for instance, that $0 < \triangle\tilde{\sigma} < \pi$, the point $\psi^{(3)}_{(1)}$ drifts along the line from $(0,0)$ to $(\triangle\tilde{\sigma}, 0)$ with the first component increasing, the point $\psi^{(3)}_{(2),0}$ drifts along the line from $(0,\pi)$ to $(\triangle\tilde{\sigma}+\pi,\pi)$ with the first component decreasing, etc. When we consider the perturbed points $\psi^{(3)}_{(j)} = \psi^{(3)}_{(j),0} + \mathcal{O}(\eta)$, these lines turn into close curves.

(3) If $\triangle\tilde{\sigma}$ is close to 0 or π, then one of the determinants, given to the first order by $E^{(\pm)}$, can be very small. Indeed, for $Q = 1/2$ one has $E^{(\pm)} = \sqrt{\frac{1}{2}(1 + (\pm 1)^\tau \cos\triangle\tilde{\sigma})}$. Then, studying more carefully the term $\mathcal{O}(\eta)$ neglected in the equations, one can show that near this value $Q = 1/2$, some of the four critical points may bifurcate. Examples of such bifurcations were given in [22].

(4) As regards the possibility of bifurcations, we emphasize here that two different situations may occur depending on whether $\tau = \mathrm{tr}U$ is odd or even. This can be seen from the expressions for $E^{(\pm)}$ given in the previous remark for $Q = 1/2$ by studying when such expressions vanish. We see that if τ is odd, then for $\triangle\tilde{\sigma}$ close to 0, the critical points $\psi^{(3)}_{(2,4)}$ may bifurcate, whereas the points $\psi(3)_{(2,4)}$ continue; and for $\triangle\tilde{\sigma}$ close to π, we have the opposite situation. On the contrary, if τ is even, then for $\triangle\tilde{\sigma}$ close to 0, the four critical points continue; and for $\triangle\tilde{\sigma}$ close to π, the four critical points may bifurcate.

Next we translate the results of Lemma 4 through the linear change (51). As follows from (53), each critical point of the function $\mathcal{K}^{(i)}(\psi)$ gives rise to κ critical points of $\mathcal{L}^{(i)}(\theta)$. For each critical point $\theta^{(i)}_*$, we also find an estimate for (the modulus of) the minimum eigenvalue $m^{(i)}_*$ of the symmetric matrix $\mathrm{D}^2\mathcal{L}^{(i)}$ at this point. This eigenvalue is closely related to the transversality of the homoclinic orbit associated with the critical point.

Lemma 5.

(a) *The function $\mathcal{L}^{(2)}$ has precisely 4κ critical points $\theta^{(2)}_*$, all nondegenerate and satisfying*

$$m^{(2)}_* \sim \sqrt{\varepsilon}\mathcal{L}_{S_2}.$$

(b) *Assuming $\tilde{\sigma}^* > 0$ and $\mathcal{L}_{S_2} \preceq \tilde{\sigma}^*\mathcal{L}_{S_1}$, the function $\mathcal{L}^{(3)}$ has precisely 4κ critical points $\theta^{(3)}_*$, all nondegenerate and satisfying*

$$\tilde{\sigma}^*\sqrt{\varepsilon}\mathcal{L}_{S_2} \preceq m^{(3)}_* \preceq \sqrt{\varepsilon}\mathcal{L}_{S_2}.$$

Proof. For the minimum (in modulus) eigenvalue of $\mathrm{D}^2\mathcal{L}^{(2)}(\theta^{(2)}_*)$, we use the following expression:

$$m^{(2)}_* = \frac{2|D|}{|T| + \sqrt{T^2 - 4D}}, \tag{57}$$

where $D = \det \mathrm{D}^2\mathcal{L}^{(2)}(\theta^{(2)}_*)$ and $T = \mathrm{tr}\mathrm{D}^2\mathcal{L}^{(2)}(\theta^{(2)}_*)$. Thus we have to find estimates for D and T. It is clear that $\mathrm{D}^2\mathcal{L}^{(2)}(\theta^{(2)}_*) = \mathcal{A}_n^\top \mathrm{D}^2\mathcal{K}^{(2)}(\psi^{(2)}_*)\mathcal{A}_n$, and, since $|\det\mathcal{A}_n| = \kappa$, it follows directly from Lemma 4 that $|D| = \kappa^2 AB = \kappa^2\mathcal{L}_{S_1}\mathcal{L}_{S_2}$. On the other hand, writing $\mathrm{D}^2\mathcal{K}^{(2)}(\psi^{(2)}_*) = \begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix}$, we see that

$$\begin{aligned}\mathrm{D}^2\mathcal{L}^{(2)}(\psi^{(2)}_*) &= k_{11}s_0(n-1)s_0(n-1)^\top \\ &\quad + k_{12}(s_0(n-1)s_0(n)^\top + s_0(n)s_0(n-1)^\top) \\ &\quad + k_{22}s_0(n)s_0(n)^\top,\end{aligned}$$

and we obtain

$$T = k_{11}\langle s_0(n-1), s_0(n-1)\rangle + 2k_{12}\langle s_0(n-1), s_0(n)\rangle + k_{22}\langle s_0(n), s_0(n)\rangle. \tag{58}$$

Using the formulas $|k_{11}| = A$, $|k_{22}| = B$, and $k_{12} = 0$, and also the formula $\mathcal{L}_{S_1} = \max(A, B)$ and estimate (46), we deduce that $|T| \sim \mathcal{L}_{S_1}/\sqrt{\varepsilon}$. Since $|D| \ll T^2$, we see from (57) that

$$m_*^{(2)} \sim \frac{|D|}{|T|} \sim \sqrt{\varepsilon}\mathcal{L}_{S_2}.$$

To estimate the minimum eigenvalue of $\mathrm{D}^2\mathcal{L}^{(3)}(\theta_*^{(3)})$, we can proceed analogously. Applying Lemma 4, we obtain

$$|D| = \kappa^2 B^2 (E^{(\pm)}\eta + \mathcal{O}(\eta^2)) \sim \mathcal{L}_{S_1}^2 E^{(\pm)}\eta \sim E^{(\pm)}\mathcal{L}_{S_1}\mathcal{L}_{S_2} \succeq \tilde{\sigma}^*\mathcal{L}_{S_1}\mathcal{L}_{S_2}$$

under the following additional condition (required in Lemma 4):

$$\eta \sim \frac{\mathcal{L}_{S_2}}{\mathcal{L}_{S_1}} \preceq \tilde{\sigma}^*.$$

We can give an estimate for T using (58) again, but now with $|k_{22}| = B(1+\mathcal{O}(\eta)) \sim \mathcal{L}_{S_1}$ and $|k_{11}|, |k_{12}| \preceq B\eta \sim \mathcal{L}_{S_2}$, obtaining the same estimate for $|T|$ as before and, consequently, the expected estimate for $m_*^{(3)}$. $\square$

After we have studied the critical points of the approximations $\mathcal{L}^{(2)}$ and $\mathcal{L}^{(3)}$, the last step is to study their persistence in the whole splitting potential $\mathcal{L}$. In order to apply the result on $\mathcal{L}^{(3)}$ from Lemma 5 and establish (for suitable quadratic frequencies) the existence and continuation of nondegenerate critical points for all $\varepsilon \to 0$, we will assume in (7) that the difference of phases

$$\triangle\sigma_n := \sigma_{s_0(n+1)} - \tau\sigma_{s_0(n)} + \delta\sigma_{s_0(n-1)}$$

keeps far away from 0 or $\pi \pmod{2\pi}$ for any n: for some fixed $\sigma^* > 0$,

$$\min(|\triangle\sigma_n|, |\triangle\sigma_n - \pi|) \geq \sigma^* \quad \forall n \geq 1. \tag{59}$$

As a specific example where condition (59) holds, we can consider in (7) the sequence of (primary) phases given by the recurrence relation $\sigma_{s_0(n+1)} = \tau\sigma_{s_0(n)} - \delta\sigma_{s_0(n-1)} + \pi/2$, $n \geq 1$, with any initial $\sigma_0(1)$ and $\sigma_0(2)$. On the contrary, we emphasize that condition (59) does not hold in the case of a reversible perturbation given by an even function $f(\varphi)$. In such a case, bifurcations of some of the homoclinic orbits for ε going across some critical values were described in [22].

The next theorem is formulated in terms of the splitting function $\mathcal{M}(0, \theta) = \partial_\theta\mathcal{L}(0, \theta)$, which gives a measure of distance between the whiskers. In this theorem, we establish, under some conditions, the existence of 4κ simple zeros of $\mathcal{M}(0, \theta)$, denoted by θ_*; and for these zeros, we provide an estimate for the minimum (in modulus) eigenvalue of the *splitting matrix* $\partial_\theta\mathcal{M}(0, \theta_*)$. As was pointed out in [4], this minimum eigenvalue provides a lower bound for the transversality of the homoclinic orbit associated with the zero θ_*. Recall that the integer $\kappa \geq 1$ was introduced in (53).

To obtain the continuation of the critical points for all $\varepsilon \to 0$, we have to assume that our quadratic frequency vector ω satisfies the *strong separation condition*

$$\sqrt{\tilde{\gamma}^{**}} > 2A_2 - 1 \tag{60}$$

(recall that the constants A_i were defined in (43)). Otherwise, we can obtain the persistence of all critical points for ε not very close to the critical values ε_n (in other words, for ε close enough to the values ε'_n) provided that ω satisfies the *weak separation condition*

$$\sqrt{\tilde{\gamma}^{**}} > A_1. \tag{61}$$

We emphasize that the two separation conditions can be checked explicitly for concrete quadratic frequencies. For instance, in the four examples considered in Sec. 2, we checked that Example 1 (the golden vector) satisfies (60), Examples 2 and 3 satisfy (61), and Example 4 satisfies none of them. Unfortunately, it seems from our numerical experiments that the only frequency vectors satisfying the strong condition (60) are the golden vector and other noble vectors (that can be reduced to the golden vector by a unimodular transformation; the constants $\tilde{\gamma}^{**}$ and A_i are the same for all of them). Nevertheless, the result obtained may be relevant if we take into account that noble vectors are dense.

For the sake of completeness, we have also included a much simpler statement concerning the maximum size (in modulus) of the splitting function $\mathcal{M}(0, \theta)$, thus giving an asymptotic estimate for the maximum splitting distance. Observe the difference in the exponents given by the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$, illustrated in Fig. 1.

Theorem 6. *In the example introduced in* (1)–(7)*, assume that* $\varepsilon \preceq 1$ *and* $\mu = \varepsilon^p$*,* $p > p^*$*, with* p^* *as defined in* (18)–(19)*. Assume also that condition* (59) *on the phases is fulfilled. Then*

(a) *Under the weak separation condition* (61)*, the following estimate holds:*

$$\max_{\theta \in \mathbb{T}^2} |\mathcal{M}(0,\theta)| \sim \frac{\mu}{\sqrt{\varepsilon}} \exp\Big\{ -\frac{C_0 h_1(\varepsilon)}{\varepsilon^{1/4}} \Big\}.$$

(b) *Under the weak separation condition* (61)*, there exists* ζ*, with* $1 < \zeta < \lambda^2$*, such that if* ε *belongs to some interval* $(\zeta \varepsilon_n, \varepsilon_{n-1}/\zeta)$*, then the function* $\mathcal{M}(0,\theta)$ *has precisely* 4κ *zeros* θ_**, all simple* (*with the integer* $\kappa \geq 1$ *defined in* (53)).

(c) *Under the strong separation condition* (60)*, for any* $\varepsilon \preceq (\sigma^*)^{1/(p-p^*)}$*, the function* $\mathcal{M}(0,\theta)$ *has precisely* 4κ *zeros* θ_**, all simple.*

In both cases (b) *and* (c)*, the minimum* (*in modulus*) *eigenvalue of* $\partial_\theta \mathcal{M}(0,\cdot)$ *at every zero satisfies the estimate*

$$\sigma^* \mu \varepsilon^{1/4} \exp\Big\{ -\frac{C_0 h_2(\varepsilon)}{\varepsilon^{1/4}} \Big\} \preceq m_* \preceq \mu \varepsilon^{1/4} \exp\Big\{ -\frac{C_0 h_2(\varepsilon)}{\varepsilon^{1/4}} \Big\}.$$

Proof. We write $\mathcal{M} = \partial_\theta \mathcal{L}$. To prove (a), we consider, as in (50), the approximation $\mathcal{L}^{(2)}$ given by the two most dominant harmonics. We can easily give an estimate for $\partial_\theta \mathcal{L}^{(2)}$ by writing it in the variables ψ, as in (54), and applying Lemma 3 (we use the notation $|\cdot|$ for the supremum norm on $\mathbb{T}^2$):

$$|\partial_\theta \mathcal{L}^{(2)}| \sim \frac{1}{\varepsilon^{1/4}} \mathcal{L}_{S_1}, \qquad |\partial_\theta \mathcal{L}(0,\cdot) - \partial_\theta \mathcal{L}^{(2)}| \preceq \frac{1}{\varepsilon^{1/4}} (\mathcal{L}_{S_3} + \mathcal{S}),$$

where we set, for the bound originating from the secondary resonances,

$$\mathcal{S} = \frac{\mu}{\varepsilon^{1/4}} \exp\Big\{ -\frac{C_0 \sqrt{\tilde{\gamma}^{**}}}{\varepsilon^{1/4}} \Big\}.$$

Since we always have $h_1(\varepsilon) < h_3(\varepsilon)$, we obtain the expected asymptotic estimate (a) for $|\mathcal{M}(0,\cdot)|$.

To prove (b) and (c), we will show that $\mathcal{L}$ has nondegenerate critical points by choosing $\mathcal{L}^{(2)}$ or $\mathcal{L}^{(3)}$ as a suitable approximation, since we know from Lemma 5 that the critical points of these functions are all nondegenerate. The choice of the approximation depends on the closeness of ε to the values ε_n and ε'_n. More precisely, we take $\mathcal{L}^{(2)}$ in some interval around ε'_n, and we take $\mathcal{L}^{(3)}$ near ε_n. We are going to find out whether the two intervals where the approximations are valid intersect and the results are valid for all ε (small enough) or, on the contrary, some intervals of ε must be excluded. This will depend on the separation $\tilde{\gamma}^{**}$.

First, for $\varepsilon \in (\varepsilon_n, \varepsilon_{n-1})$, we consider the first approximation for the function $G(\theta) = \mathcal{M}(0,\theta) = \partial_\theta \mathcal{L}(0,\theta)$ given by $G_0(\theta) = \partial_\theta \mathcal{L}^{(2)}(\theta)$. Recall that the zeros $\theta_*^{(2)}$ of G_0 are all simple and an estimate for $m_*^{(2)}$, the minimum eigenvalue of $\mathrm{D}G_0(\theta_*^{(2)})$, was given in Lemma 5(a). According to the quantitative version of the implicit function theorem (see, for instance, [6, Appendix]), this zero of G_0 persists as a (perturbed) zero θ_* of G, provided that inequalities of the following types are satisfied:

$$|G - G_0| \preceq \frac{(m_*^{(2)})^2}{|\mathrm{D}^2 G_0|}, \qquad |\mathrm{D}G - \mathrm{D}G_0| \preceq m_*^{(2)}. \tag{62}$$

Lemmas 3(b) and 5(a) imply the estimates

$$|G - G_0| \preceq \frac{1}{\varepsilon^{1/4}} (\mathcal{L}_{S_3} + \mathcal{S}), \qquad |\mathrm{D}G - \mathrm{D}G_0| \preceq \frac{1}{\varepsilon^{1/2}} (\mathcal{L}_{S_3} + \mathcal{S}),$$
$$m_*^{(2)} \sim \sqrt{\varepsilon} \mathcal{L}_{S_2}, \qquad |\mathrm{D}^2 G_0| \preceq \frac{1}{\varepsilon^{3/4}} \mathcal{L}_{S_1}.$$

Using these estimates, we see from (62) that we need the relation

$$\mathcal{L}_{S_3} + \mathcal{S} \preceq \frac{\varepsilon^2 \mathcal{L}_{S_2}^2}{\mathcal{L}_{S_1}}. \tag{63}$$

Taking logarithms, we see that (63) can be written as the following inequality (with a suitable constant c):

$$2h_2(\varepsilon) - h_1(\varepsilon) \le h_3^*(\varepsilon) + \frac{\varepsilon^{1/4}}{C_0}\log(c\varepsilon^2), \quad h_3^*(\varepsilon) := \min(h_3(\varepsilon), \sqrt{\tilde{\gamma}^{**}}). \tag{64}$$

This inequality will be analyzed later.

For $\varepsilon \in (\varepsilon'_{n+1}, \varepsilon'_n)$, we consider the first approximation for $G(\theta)$ given by $G_0(\theta) = \partial_\theta \mathcal{L}^{(3)}(\theta)$. In this case, in order to apply Lemma 5(b) to the zeros $\theta_*^{(3)}$ of G_0, we have to check two additional conditions. The first one is that the difference of phases $\triangle\tilde{\sigma}_n = \tilde{\sigma}_{s_0(n+1)} - \tau\tilde{\sigma}_{s_0(n)} + \delta\tilde{\sigma}_{s_0(n-1)}$ is not very close to 0 or π (mod 2π). Indeed, it follows from Lemma 3(a) that $|\triangle\tilde{\sigma}_n - \triangle\sigma_n| \preceq \mu\varepsilon^{-p^*}$. Recalling that $\mu = \varepsilon^p$, we see that $\mu\varepsilon^{-p^*} \preceq \sigma^*$, and (59) yields the lower bound $\min(|\triangle\tilde{\sigma}_n|, |\triangle\tilde{\sigma}_n - \pi|) \succeq \sigma^*$. The second condition is $\mathcal{L}_{S_2} \preceq \sigma^* \mathcal{L}_{S_1}$, which can be written as

$$h_1(\varepsilon) \le h_2(\varepsilon) + \frac{\varepsilon^{1/4}}{C_0}\log(c\sigma^*). \tag{65}$$

Lemma 5(b) gives a lower bound for the minimum eigenvalue $m_*^{(3)}$. Applying the implicit function theorem as before, we see that the zero $\theta_*^{(3)}$ persists as a zero θ_* of G provided that

$$\mathcal{L}_{S_4} + \mathcal{S} \preceq \frac{(\sigma^*\varepsilon)^2 \mathcal{L}_{S_2}^2}{\mathcal{L}_{S_1}},$$

which can be written as

$$2h_2(\varepsilon) - h_1(\varepsilon) \le h_4^*(\varepsilon) + \frac{\varepsilon^{1/4}}{C_0}\log(c(\sigma^*\varepsilon)^2), \quad h_4^*(\varepsilon) := \min(h_4(\varepsilon), \sqrt{\tilde{\gamma}^{**}}). \tag{66}$$

To obtain the continuation of the perturbed zeros, it is necessary that every ε satisfy (64) or (65)–(66). To study this condition, we may restrict ourselves to the interval $[\varepsilon_n, \varepsilon'_n]$ (the intersection of the two intervals considered above), and the results can easily be extended outside this interval by using the symmetries of the functions $h_i(\varepsilon)$.

The behavior of the involved functions on the interval $[\varepsilon_n, \varepsilon'_n]$ can be described as follows (see also Fig. 1): the function h_1 increases from 1 to A_1, the function h_2 decreases from A_2 to A_1, the function h_3 increases from A_2 to A_3, the function h_4 decreases from A_4 to A_3, and the function $2h_2 - h_1$ decreases from $2A_2 - 1$ to A_1. In addition, the functions h_3 and $2h_2 - h_1$ intersect at some point $\bar{\varepsilon}_n$ with a common value $\bar{A}$ (the values $\bar{\varepsilon}_n$ and $\bar{A}$ can be found explicitly if desired). Recall that the constants A_i defined in (43) depend only on $|\lambda| > 1$. One can check that $A_2 < \bar{A} < 2A_2 - 1 < A_3$ (to check the last inequality, one may use the fact that $|\lambda|^{3/2}(A_3 - 2A_2 + 1)$ is a polynomial in $|\lambda|^{1/2}$ with no real roots for $|\lambda| > 1$).

To study the intervals where inequalities (64)–(66) are satisfied, we have to replace h_3 and h_4 by h_3^* and h_4^*. Hence we have to take into account the position of $\sqrt{\tilde{\gamma}^{**}}$ with respect to the values A_i, $2A_2 - 1$, and $\bar{A}$ considered above. Moreover, there is a contribution originating from the term containing $\varepsilon^{1/4}$ and the logarithm in the three inequalities. This small term gives rise only to a small perturbation of the results, although it has to be seriously taken into account in inequality (65), which will be true for all $\varepsilon \in [\varepsilon_n, \varepsilon'_n)$ excluding a small neighborhood of ε'_n. On the other hand, note that (64) always implies (66), the approximation given by the two dominant harmonics is valid, and the approximation given by the three dominant harmonics is valid as well unless ε belongs to the small neighborhood where inequality (65) does not hold.

Taking these considerations into account, we can check that if the strong separation condition (60) is satisfied, then every ε in the whole interval under consideration satisfies (64) or (65)–(66), and hence the zeros persist. More precisely, we can consider two dominant harmonics only for $\varepsilon \in (\bar{\varepsilon}_n, \varepsilon'_n]$, and three dominant harmonics for $\varepsilon \in [\varepsilon_n, \bar{\varepsilon}_n]$.

If the strong condition does not work, but instead the weak condition (61) is fulfilled, then there exists ζ with $1 < \zeta < \lambda^2$ such that every $\varepsilon \in (\zeta\varepsilon_n, \varepsilon'_n]$ satisfies (64) or (65)–(66). The value $\zeta\varepsilon_n$ is the solution of the equation $2h_2(\varepsilon) - h_1(\varepsilon) = \sqrt{\tilde{\gamma}^{**}}$; it moves from ε_n to ε'_n as $\sqrt{\tilde{\gamma}^{**}}$ goes down from $2A_2 - 1$ to A_1 (eventually, the interval shrinks to ε'_n). If the weak condition (61) is not satisfied, then the interval is empty and both inequalities (64) and (66) are false. Note that the interval $(\zeta\varepsilon_n, \varepsilon'_n]$ turns into $(\zeta\varepsilon_n, \varepsilon_{n-1}/\zeta)$ if we extend the results using the symmetry of the functions h_i (regarded as functions of $\log\varepsilon$).

For the cases where the persistence of the zeros has been established, the upper and lower bounds for the minimum eigenvalue at each zero result from Lemma 5. $\square$

Remarks.

(1) In the weak condition case considered in (b), it can be seen from the proof that for $\sqrt{\tilde{\gamma}^{**}} \le \bar{A}$, condition (59) can be removed, because we do not need to consider the case of three dominant harmonics. We have ignored this in order to give a simpler statement of the theorem.

(2) As mentioned in Remark 3 after Lemma 3, if in (7) we consider a perturbation $f(\varphi)$ having only primary harmonics, then secondary harmonics pose no obstacles, and the result of (c) is valid independently of the separation $\tilde{\gamma}^{**}$. However, this example would be rather artificial.

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