A Methodology for Obtaining Asymptotic Estimates for the Exponentially Small Splitting of Separatrices to Whiskered Tori with Quadratic Frequencies

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1 Introduction

The aim of this work is to provide asymptotic estimates for the splitting of separatrices in a perturbed 3-degree-of-freedom Hamiltonian system, associated to a two-dimensional whiskered torus (invariant hyperbolic torus) whose frequency ratio is a quadratic irrational number. We show that the dependence of the asymptotic estimates on the perturbation parameter is described by some functions which satisfy a periodicity property, and whose behavior depends strongly on the arithmetic properties of the frequencies.

First, we describe the Hamiltonian system to be studied. It is also considered in [6], as a generalization of the famous Arnold's example [1], and provides a model for the behavior of a nearly-integrable Hamiltonian system in the vicinity of a single resonance (see [4] for a motivation). In canonical coordinates $(x, y, \varphi, I) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}^2 \times \mathbb{R}^2$, we consider a perturbed Hamiltonian

$$H(x, y, \varphi, I) = H_0(x, y, I) + \mu H_1(x, \varphi). \tag{1}$$

$$H_0(x, y, I) = \langle \omega_\varepsilon, I \rangle + \frac{1}{2}\langle \Lambda I, I \rangle + \frac{y^2}{2} + \cos x - 1, \tag{2}$$

$$H_1(x, \varphi) = \cos x \cdot \sum_{k_2 \geq 0} \mathrm{e}^{-\rho|k|} \cos(\langle k, \varphi \rangle - \sigma_k). \tag{3}$$

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For the integrable Hamiltonian H_0, we consider a vector of *fast frequencies*

$$\omega_\varepsilon = \frac{\omega}{\sqrt{\varepsilon}}, \qquad \omega = (1, \Omega), \tag{4}$$

where the frequency ratio Ω is a *quadratic* irrational number. In this way, our system has two parameters $\varepsilon > 0$ and μ, but we assume them linked by a relation of the kind $\mu = \varepsilon^p$, $p > 0$ (the smaller p the better). Thus, if we consider ε as the unique parameter, we have a *singular* or *weakly hyperbolic* problem for $\varepsilon \to 0$ (see [4] for a discussion about singular and regular problems).

On the other hand, notice that H_0 consists of a classical pendulum and two rotors (in the coordinates x, y and φ, I respectively). Then, we see that H_0 has a family of two-dimensional whiskered tori, with coincident whiskers (invariant manifolds). Such tori can be indexed by the (constant) action I, and have frequency vectors $\omega_\varepsilon + \Lambda I$. We assume that the matrix Λ is such that the condition of *isoenergetic nondegeneracy* is satisfied (see, for instance, [6]). Among the tori, we fix our attention on the torus given by $I = 0$,

$$\mathcal{T}_0: \quad (0, 0, \theta, 0), \quad \theta \in \mathbb{T}^2,$$

whose inner flow is given, in this parameterization, by $\dot\theta = \omega_\varepsilon$. This torus has a homoclinic whisker (i.e., coincident stable and unstable whiskers),

$$\mathcal{W}_0: \quad (x_0(s), y_0(s), \theta, 0), \quad s \in \mathbb{R},\ \theta \in \mathbb{T}^2,$$

where $x_0(s) = 4 \arctan \mathrm{e}^s$, $y_0(s) = 2/\cosh s$ (the upper separatrix of the classical pendulum). The inner flow on $\mathcal{W}_0$ is given by $\dot s = 1$, $\dot\theta = \omega_\varepsilon$.

Concerning the perturbation H_1, it is given by a constant $\rho > 0$ (the complex width of analyticity in the angles φ), and phases σ_k that, for the purpose of this work, can be chosen arbitrarily.

Under the hypotheses described, the *hyperbolic KAM theorem* (see, for instance, [8]) can be applied to the perturbed Hamiltonian (1)–(3). We have that, for $\mu \neq 0$ small enough, the whiskered torus $\mathcal{T}_0$ persists with some shift and deformation giving rise to a perturbed torus $\mathcal{T}$, with perturbed local stable and unstable whiskers.

Such local whiskers can be extended to global whiskers $\mathcal{W}^{\mathrm{s}}$, $\mathcal{W}^{\mathrm{u}}$ but, in general, for $\mu \neq 0$ they do not coincide anymore, and one can introduce a *splitting function* giving the distance between the whiskers in the directions of the actions $I \in \mathbb{R}^2$: denoting $\mathcal{J}^{\mathrm{s,u}}(\theta)$ parameterizations of a transverse section of both whiskers, one can define $\mathcal{M}(\theta) := \mathcal{J}^{\mathrm{u}}(\theta) - \mathcal{J}^{\mathrm{s}}(\theta)$, $\theta \in \mathbb{T}^2$. In fact, this function turns out to be the gradient of the (scalar) *splitting potential*: $\mathcal{M}(\theta) = \nabla \mathcal{L}(\theta)$ (see [3, Sect. 5.2], and also [7]).

In (4), we deal with the following 24 quadratic numbers

$$[\overline{1}], [\overline{2}], \ \ldots, \ [\overline{13}], [\overline{1,2}], \ \ldots, \ [\overline{1,12}], \tag{5}$$

where we denote a quadratic number according to its periodic part in the continued fraction, see (8).

Next, we establish the *main result* of this work, providing two types of *asymptotic estimates* for the splitting, as $\varepsilon \to 0$. On one hand, we give an estimate for the *maximal splitting distance*, i.e., for the maximum of $|\mathcal{M}(\theta)|$, $\theta \in \mathbb{T}^2$. On the other hand, we show that for most values of $\varepsilon \to 0$ there exist four transverse homoclinic orbits, associated to simple zeros θ_* of $\mathcal{M}(\theta)$ (i.e., nondegenerate critical points of $\mathcal{L}(\theta)$) and, for such homoclinic orbits, we obtain an estimate for the *transversality* of the splitting, given by the minimum eigenvalue (in modulus) of the matrices $\mathrm{D}\mathcal{M}(\theta_*)$.

We use the notation $f \sim g$ if we can bound $c_1|g| \le |f| \le c_2|g|$ with positive constants c_1, c_2 not depending on ε, μ.

Theorem 1 *Assume the conditions described above for the Hamiltonian* (1)–(3)*, and that* ε *is small enough and* $\mu = \varepsilon^p$, $p > 3$. *Then, there exist continued functions* $h_1(\varepsilon)$ *and* $h_2(\varepsilon)$ *(defined in* (17)*), periodic in* $\ln \varepsilon$ *and satisfying* $1 \le h_1(\varepsilon) \le h_2(\varepsilon)$*, and a positive constant* C_0 *(given in* (16)*), such that:*

(i) *for the maximal splitting distance, we have the estimate*

$$\max_{\theta \in \mathbb{T}^2} |\mathcal{M}(\theta)| \sim \frac{\mu}{\sqrt{\varepsilon}} \exp\left\{-\frac{C_0 h_1(\varepsilon)}{\varepsilon^{1/4}}\right\} ;$$

(ii) *the splitting function* $\mathcal{M}(\theta)$ *has exactly four zeros* θ_**, all simple, for all* ε *except for a small neighborhood of a finite number of geometric sequences of* ε*;*

(iii) *at each zero* θ_* *of* $\mathcal{M}(\theta)$*, the minimal eigenvalue of* $\mathrm{D}\mathcal{M}(\theta_*)$ *satisfies*

$$m_* \sim \mu \varepsilon^{1/4} \exp\left\{-\frac{C_0 h_2(\varepsilon)}{\varepsilon^{1/4}}\right\} .$$

For the proof of this theorem, we apply the *Poincaré–Melnikov method*, which provides a first order approximation

$$\mathcal{M}(\theta) = \mu \nabla L(\theta) + \mathcal{O}(\mu^2) \tag{6}$$

in terms of the *Melnikov potential*, which can be defined by integrating the perturbation H_1 along the trajectories of the unperturbed homoclinic whisker $\mathcal{W}_0$:

$$L(\theta) := -\int_{-\infty}^{\infty} H_1(x_0(t), \theta + \omega_\varepsilon t)\, \mathrm{d}t. \tag{7}$$

Since this first order approximation is exponentially small in ε, in principle the approximation (6) cannot be directly applied in our singular problem with $\mu = \varepsilon^p$. However, using suitable bounds for the error term $\mathcal{O}(\mu^2)$, given in [6], one can see that for $p > 3$ the first order approximation given by the Melnikov potential overcomes the error term and provides the right asymptotic estimates for the splitting. Such estimates come from the size of *dominant harmonics* in the Fourier expansion of (7), and studying their dependence on ε. More precisely, to estimate the maximal splitting one dominant harmonic is enough and, to estimate the transversality of the splitting, two dominant harmonics are required (excluding the values of ε such that the second and third harmonics are of the same magnitude, which could give rise to bifurcations in the homoclinic orbits and would require a further study).

The remaining sections of this work are devoted to the definition of the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$, making emphasis on their dependence on the arithmetic properties of the quadratic number Ω.

2 Continued Fractions and Resonant Sequences

We review briefly the technique developed in [5] for studying the resonances of quadratic frequencies. Let $0 < \Omega < 1$ be a quadratic irrational number. It is well-known that it has an infinite continued fraction

$$\Omega = [a_1, a_2, a_3, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}, \qquad a_n \in \mathbb{Z}^+,\ n \geq 1 \quad (\text{and } a_0 = 0), \tag{8}$$

which is *eventually periodic*, i.e., periodic starting at some a_l. For a purely m-periodic continued fraction $\Omega = [\overline{a_1, \ldots, a_m}]$ we introduce the matrix

$$U = (-1)^m A_1^{-1} \cdots A_m^{-1}, \text{ where } A_l = \begin{pmatrix} a_l & 1 \\ 1 & 0 \end{pmatrix}, \quad l = 1, \ldots, m.$$

It is well-known that quadratic vectors satisfy a *Diophantine condition*

$$|\langle k, \omega \rangle| \geq \frac{\gamma}{|k|}, \quad \forall k \in \mathbb{Z}^2 \setminus \{0\}.$$

With this in mind, we define the *"numerators"*

$$\gamma_k := |\langle k, \omega \rangle| \cdot |k|, \qquad k \in \mathbb{Z}^2 \setminus \{0\} \tag{9}$$

(for integer vectors, we use the norm $|\cdot| = |\cdot|_1$). Our aim is to find the integer vectors k which give the smallest values γ_k; we call such vectors the *primary resonances*.

All vectors $k \in \mathbb{Z}^2 \setminus \{0\}$ with $|\langle k, \omega \rangle| < 1/2$ are subdivided into *resonant sequences*:

$$s(j, n) := U^n k^0(j), \qquad n = 0, 1, 2, \ldots, \tag{10}$$

where the initial vector $k^0(j) = (-\operatorname{rint}(j\Omega), j)$, $j \in \mathbb{Z}^+$, satisfies

$$\frac{1}{2\lambda} < |\langle k^0(j), \omega\rangle| < \frac{1}{2}, \tag{11}$$

λ being the eigenvalue of U with $\lambda > 1$. For each $j \in \mathbb{Z}^+$ satisfying (11), it was proved in [5, Theorem 2] (see also [2]) that, asymptotically, the resonant sequence $s(j,n)$ exhibits a geometric growth and the sequence $\gamma_{s(j,n)}$ has a limit γ_j^*:

$$|s(j,n)| = K_j\lambda^n + \mathcal{O}(\lambda^{-n}), \quad \gamma_{s(j,n)} = \gamma_j^* + \mathcal{O}(\lambda^{-2n}), \text{ as } n \to \infty, \tag{12}$$

where K_j and γ_j^* can be determined explicitly for each resonant sequence (see explicit formulas in [5]). We select the minimal of γ_j^*:

$$\gamma^* := \liminf_{|k|\to\infty} \gamma_k = \min_j \gamma_j^* = \gamma_{j_0}^* > 0. \tag{13}$$

The integer vectors of the corresponding sequence $s(j_0, n)$ are *the primary resonances*, and we call the *secondary resonances* the integer vectors belonging to any of the remaining resonant sequences $s(j,n)$, $j \neq j_0$. We also call by *the main secondary resonances* the sequence $s(j_1, n)$ which is linearly independent with $s(j_0, n)$ and gives the smallest limit $\gamma_{j_1}^*$ among the secondary resonances.

3 The Functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$

Taking into account the form of H_1 in (3), we present the Melnikov potential (7) in its Fourier expansion. Using (4) and (9), we present the coefficients in the form

$$L_k = \frac{2\pi\,|\langle k, \omega_\varepsilon\rangle|\,\mathrm{e}^{-\rho|k|}}{\sinh|\frac{\pi}{2}\langle k, \omega_\varepsilon\rangle|} = \alpha_k\,\mathrm{e}^{-\beta_k}, \qquad \alpha_k \approx \frac{4\pi\gamma_k}{|k|\sqrt{\varepsilon}}, \quad \beta_k = \rho|k| + \frac{\pi\gamma_k}{2|k|\sqrt{\varepsilon}}. \tag{14}$$

For any given ε, we find the dominant harmonics $L_k(\varepsilon)$ which correspond essentially to the smallest exponents $\beta_k(\varepsilon)$.

The exponents $\beta_k(\varepsilon)$ in (14) can be presented in the form

$$\beta_k(\varepsilon) = \frac{C_0}{\varepsilon^{1/4}}\, g_k(\varepsilon), \qquad g_k(\varepsilon) := \frac{\tilde{\gamma}_k^{1/2}}{2}\left[\left(\frac{\varepsilon}{\varepsilon_k}\right)^{1/4} + \left(\frac{\varepsilon_k}{\varepsilon}\right)^{1/4}\right], \tag{15}$$

where

$$\varepsilon_k := D_0\,\frac{\tilde{\gamma}_k^{\,2}}{|k|^4}, \qquad \tilde{\gamma}_k = \frac{\gamma_k}{\gamma^*}, \qquad C_0 = (2\pi\rho\gamma^*)^{1/2}, \qquad D_0 = \left(\frac{\pi\gamma^*}{2\rho}\right)^2. \tag{16}$$

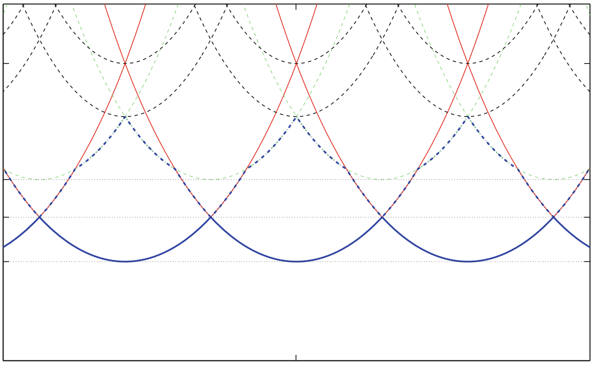

Fig. 1 Graphs of the functions $h_1(\varepsilon)$ (*solid blue*) and $h_2(\varepsilon)$ (*dash-dot blue*) for $[\overline{1,2}] = \sqrt{3} - 1$. *Red lines* are the primary functions $g_{s(j_0,n)}(\varepsilon)$, and *green lines* correspond to the main secondary functions $g_{s(j_1,n)}(\varepsilon)$

Since the coefficients L_k are exponentially small in ε, it is more convenient to work with the functions g_k, whose smallest values correspond to the largest L_k. To this aim, it is useful to consider the graphs of the functions $g_k(\varepsilon)$, $k \in \mathbb{Z}^2 \setminus \{0\}$, in order to detect the minimum of them for a given value of ε.

We know from (15) that the functions $g_k(\varepsilon)$ have their minimum at $\varepsilon = \varepsilon_k$ and the corresponding minimal values are $g_k(\varepsilon_k) = \tilde{\gamma}_k^{1/2}$. For the integer vectors $k = s(j, n)$ belonging to a resonant sequence (10), using the approximations (12), we have

$$\varepsilon_{s(j,n)} \approx \frac{D_0(\tilde{\gamma}_j^*)^2}{K_j^4 \lambda^{4n}}, \quad g_{s(j,n)}(\varepsilon) \approx \frac{(\tilde{\gamma}_j^*)^{1/2}}{2} \left[\left(\frac{\varepsilon}{\varepsilon_{s(j,n)}} \right)^{1/4} + \left(\frac{\varepsilon_{s(j,n)}}{\varepsilon}^{1/4} \right) \right], \text{ as } n \to \infty.$$

Taking into account such approximations, we have a periodic behavior of the functions with respect to $\ln \varepsilon$, as we see in Fig. 1 (where a logarithmic scale for ε is used).

We define, for any given ε, the function $h_1(\varepsilon)$ and $h_2(\varepsilon)$ as

$$h_1(\varepsilon) := \min_k g_k(\varepsilon) = g_{S_1}(\varepsilon), \quad h_2(\varepsilon) := \min_{k \text{ lin.indep.of } S_1} g_k(\varepsilon) = g_{S_2}(\varepsilon), \tag{17}$$

with some integer vectors $S_1(\varepsilon)$ and $S_2(\varepsilon)$ realizing such minima. The functions are continuous and $4 \ln \lambda$-periodic in $\ln \varepsilon$. It turns out that for the 24 quadratic numbers (5), the integer vector $S_1(\varepsilon)$ providing $h_1(\varepsilon)$ always corresponds to a primary resonance, defined in (13). On the other hand, the vector $S_2(\varepsilon)$ providing $h_2(\varepsilon)$ may correspond to primary or main secondary resonances in different intervals of ε (see Fig. 1 for an illustration for the number $[\overline{1,2}] = \sqrt{3} - 1$). There is a finite number of geometric sequences of ε, where a change in $S_2(\varepsilon)$ occurs. These points require a special study for the transversality and they are excluded in Theorem 1.

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