The theory of singular integral operators was developed by Calderón and Zygmund in the 1950s, and since then it has become a central topic of study in harmonic analysis, with many applications in both pure and applied mathematics. Two relevant examples of such operators are the Cauchy and Riesz transforms, which play a fundamental role in complex and harmonic analysis, and which have plenty of applications in PDE’s, geometric measure theory, and mathematical physics. In this book, some topics concerning the Cauchy and Riesz transforms and other singular integrals are studied from the geometric analysis point of view. Most of these topics are connected to interesting open problems, such as the relation between Riesz transforms and rectifiability. Geometric properties and applications of some capacities defined in terms of the Cauchy and Riesz transforms are also studied. This book, which is the publication of the author’s PhD dissertation, is suitable for graduate students and researchers in mathematics interested in Calderón-Zygmund theory and geometric analysis.

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Introduction

The topics covered in this work belong to the area of geometric analysis, which can be considered an interface between harmonic analysis and geometric measure theory. Most of these topics are related to interesting open problems which have been studied recently by many international mathematicians. More precisely, they are concerned with the Cauchy and Riesz transforms, two fundamental operators in harmonic analysis (in particular in Calderón-Zygmund theory), PDE’s, and geometric measure theory.

Given a positive Borel measure $\mu$ in $\mathbb{R}^d$, one way to define the $n$-dimensional Riesz transform of $f \in L^1(\mu)$ is by $R^\mu f(x) = \lim_{\epsilon \searrow 0} R^\mu_\epsilon f(x)$ (whenever the limit exists), where $x \in \mathbb{R}^d$ and

$$R^\mu_\epsilon f(x) = \int_{|x-y|>\epsilon} \frac{x-y}{|x-y|^{n+1}} f(y) \, d\mu(y)$$

denotes the truncation of the Riesz transform at level $\epsilon > 0$. We usually refer to the coordinates of this operator as the Riesz transforms of $f$ with respect to $\mu$. When $d = 2$ (i.e., $\mu$ is a Borel measure in $\mathbb{C}$), one defines the Cauchy transform of $f \in L^1(\mu)$ by $C^\mu f(x) = \lim_{\epsilon \searrow 0} C^\mu_\epsilon f(x)$ (whenever the limit exists), where $x \in \mathbb{C}$ and

$$C^\mu_\epsilon f(x) = \int_{|x-y|>\epsilon} \frac{f(y)}{x-y} \, d\mu(y)$$

(observe that $x$, $y$, and $f(y)$ are complex numbers). Usually, to avoid the problem of existence of the preceding limits, one considers the maximal operators $R^\mu_\epsilon f(x) = \sup_{\epsilon > 0} |R^\mu_\epsilon f(x)|$ and $C^\mu_\epsilon f(x) = \sup_{\epsilon > 0} |C^\mu_\epsilon f(x)|$. Notice that the Cauchy transform coincides with the 1-dimensional Riesz transform in the plane, modulo conjugation. The Cauchy and Riesz transforms are two very important examples of singular integral operators with a Calderón-Zygmund kernel. Namely, given a Borel measure $\mu$ in $\mathbb{R}^d$, $\epsilon > 0$, $x \in \mathbb{R}^d$, and $f \in L^1(\mu)$, one considers operators of the form

$$T^\mu_\epsilon f(x) = \int_{|x-y|>\epsilon} K(x-y) f(y) \, d\mu(y),$$

where the kernels $K: \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ under consideration satisfy

$$|K(x)| \leq \frac{C}{|x|^n}, \quad |\partial_x K(x)| \leq \frac{C}{|x|^{n+1}}, \quad \text{and} \quad |\partial_x \partial_x K(x)| \leq \frac{C}{|x|^{n+2}}.$$
for all $1 \leq i, j \leq d$ and $x = (x^1, \ldots, x^d) \in \mathbb{R}^d \setminus \{0\}$, where $0 < n < d$ are some fixed integers and $C > 0$ is some constant; and moreover $K(-x) = -K(x)$ for all $x \neq 0$ (i.e. $K$ is odd). We have to mention that the estimate on the second derivatives of $K$ is not a standard assumption in Calderón-Zygmund theory, but it will be a key fact in our results. Notice that the $n$-dimensional Riesz transform corresponds to the vector kernel $((x^1, \ldots, x^d)/|x|^{n+1}$, and the Cauchy transform to $(x^1, -x^2)/|x|^2$ (so, one may consider $K$ to be any scalar component of these vector kernels).

As said above, the Riesz transforms are the underlying topic of study of all the problems covered in this work, which are listed below. A brief description of the obtained results is provided. The different problems will be treated in different chapters of this book and they will be contextualized in the corresponding chapters.

Topics covered in this book: Statement and main results

Chapter 1

Failure of rational approximation on some Cantor type sets

Consider a compact set $K$ of the complex plane. Let $A(K)$ be the algebra of continuous functions on $K$ which are analytic on the interior of $K$, and $R(K)$ the closure (with the uniform convergence on $K$) of the functions that are analytic on a neighborhood of $K$. In the 60’s, A. Vitushkin gave a description of the compact sets $K$ for which $R(K) = A(K)$ in terms of the so-called analytic capacity (see [Vt]), but there is still no characterization of those compact sets in a geometric way. In this direction, Anthony G. O’Farrell raised the following question:

Question. Let $K_1$ and $K_2$ be two compact subsets of $[0,1]$ and define $K = (K_1 \times [0,1]) \cup ([0,1] \times K_2) \subseteq \mathbb{C}$. Is it true that $R(K) = A(K)$?

The compact sets of this form are commonly called gratings. It is known that the identity between these algebras holds if one of the compact sets $K_1$ or $K_2$ has no interior. However, it was not known whether the identity holds or not in general. We will provide an example of a compact set $K$ which gives a negative answer to the question, and we will also give a proof of the equality of the algebras when $K_1$ or $K_2$ has no interior. More precisely, we will show (see Theorems 1.0.2 and 1.2.2)

Theorem. Given two compact sets $K_1, K_2 \subseteq [0,1]$, we set $K := (K_1 \times [0,1]) \cup ([0,1] \times K_2) \subseteq \mathbb{C}$. If $K_1$ or $K_2$ have empty interior, then $R(K) = A(K)$. On the contrary, there exist compact sets $K_1, K_2 \subseteq [0,1]$ such that $R(K) \neq A(K)$. 

Let us mention that our proof of these facts rely heavily on a theorem of A. Vitushkin, which connects problems of rational approximation with local estimates on the analytic and continuous analytic capacities of the complement of a given compact set. Recall that, for a compact set \( F \subseteq \mathbb{C} \), the analytic capacity \( \gamma \) of \( F \) is defined by 
\[
\gamma(F) := \sup |f'(\infty)|, \quad \text{where the supremum is taken over all functions } f : \mathbb{C} \to \mathbb{C} \text{ which are analytic on } \mathbb{C} \setminus F, \text{ and with } |f| \text{ uniformly bounded by 1 on } \mathbb{C} \text{ (by definition, } f'(\infty) = \lim_{z \to \infty} z(f(z) - f(\infty))).
\]
A function \( f \) satisfying these properties is said to be *admissible* for \( \gamma(F) \). The continuous analytic capacity \( \alpha \) is defined as \( \gamma \) but one also requires the admissible functions to be continuous on \( \mathbb{C} \). There have been many recent advances in the study of the analytic capacities. An important one is the solution of Vitushkin’s conjecture by G. David (see [Da2]), which states that a compact set in the plane with finite and positive length is purely unrectifiable if and only if it has vanishing analytic capacity (one says that a set is purely unrectifiable if it intersects any rectifiable curve in a set of zero length). We also want to emphasize the result of X. Tolsa on Painlevé’s problem (that is, to find a characterization of the analytic capacity in geometric-measure theoretic terms) and the proof of the countable semiadditivity of \( \gamma \), and also the analogous results for the continuous analytic capacity (see [To7] and [To8]).

In the definitions of \( \gamma \) and \( \alpha \) one can also require the defining functions \( f \) to vanish at infinity, and since they are analytic outside \( F \), one can show that 
\[
f = \frac{1}{\pi z} * (\partial f) \quad \text{and} \quad f'(\infty) = \langle \partial f, 1 \rangle,
\]
where \( \partial f \) is taken in the sense of distributions, and \( \langle \cdot, \cdot \rangle \) stands for the natural pairing between compactly supported distributions and test functions. Hence, one can redefine the continuous analytic capacity as 
\[
\alpha(F) = \sup |\langle T, 1 \rangle|, \quad \text{where now the supremum is taken over all distributions } T \text{ such that } \text{supp}(T) \subset F \text{ and } 1/z * T \text{ is a continuous function on } \mathbb{C} \text{ and uniformly bounded by 1 everywhere.}
\]
Notice that \( 1/z * T \) corresponds to the Cauchy transform of \( T \), in the sense of distributions. Thus, this restatement of \( \alpha \) connects, through Vitushkin’s theorem, the problem posed by A. O’Farrell to the Cauchy transform.

Some comments about the question above are in order. A classical example of a compact set \( K \) such that \( R(K) \neq A(K) \) is the so-called *Swiss cheese*, which is constructed by subtracting from a fixed closed disc \( D \) of radius 1 a sequence of open discs \( D_j \subset D \) of radius \( d_j > 0 \) and with disjoint interior in a such manner that \( \sum_j d_j < \infty \) and \( K := D \setminus (\bigcup_j D_j) \) has empty interior. Then, the finite measure \( \mu := dz|_{\partial D} - \sum_j dz|_{\partial D_j} \) can be used to show that 
\[
R(K) \neq A(K).
\]
Basically, one checks that \( \mu \neq 0 \) (notice that, if \( \|\mu\| \) denotes the variation norm of \( \mu \), \( \|\mu\| = 2\pi(1 + \sum_j d_j) \)) and that, if \( q \) is a rational function with poles off \( K \), then 
\[
\int q \, d\mu = 0,
\]
so the algebras cannot coincide because otherwise \( \mu = 0 \), since \( \text{int} K = \emptyset \) and then \( A(K) = C(K) \). Thus, to produce such a compact set \( K \), it is a key fact that \( \sum_j d_j < \infty \) and that \( K \) has empty interior. In the case of gratings, this trick is no longer applicable. One would like to construct a compact set \( K \subset [0, 1]^2 \) with an infinite sequence of holes because...
one would like to have a compact set with empty interior, and a well known theorem states
that \( R(K) = A(K) \) for any compact set whose complement has finitely many components.
This infinite sequence of holes of \( K \) would produce an infinite sequence of holes of \( K_1 \), say
\( H_j^1 \), and an infinite sequence of holes of \( K_2 \), say \( H_j^2 \), for \( j \geq 0 \). Then, any measure of the form
\[ \mu := d\mu_{\partial[0,1]^2} - \sum_{i,j} d\mu_{\partial H_i^1 \times H_j^2} \] would not be finite, because for any fixed \( i \) one would have
infinite holes \( H_i^1 \times H_j^2 \) with diameter bigger than a fixed constant. This kind of phenomenon
does not allow us to apply the classical techniques to produce counterexamples in problems
of rational approximation. Moreover, the fact that the question was known to have an affir-
mative answer if \( K_1 \) or \( K_2 \) have empty interior suggested that it might have an affirmative
answer in general.

From the reader’s point of view, this chapter can be read independently. It has its own
notation and definitions, and the proofs do not involve the results of the other chapters in
this book.

**Chapter 2**

**A dual characterization of the \( C^1 \) harmonic capacity**

In order to deal with problems of harmonic approximation in the \( C^1 \)-norm, in the 90’s P.
Paramonov introduced the Lipschitz and \( C^1 \) harmonic capacities \( \kappa \) and \( \kappa_c \), respectively. Given
a bounded set \( E \subset \mathbb{R}^d \) (\( d \geq 2 \)), the Lipschitz and \( C^1 \) harmonic capacities of \( E \) are defined by
\[
\kappa(E) = \text{sup}\{ \langle 1, \Delta \varphi \rangle : \varphi \in \text{Lip}(\mathbb{R}^d), \text{supp} \Delta \varphi \subset E, \nabla \varphi(\infty) = 0, \|\nabla \varphi\|_{\infty} \leq 1 \}, \\
\kappa_c(E) = \text{sup}\{ \langle 1, \Delta \varphi \rangle : \varphi \in C^1(\mathbb{R}^d), \text{supp} \Delta \varphi \subset E, \nabla \varphi(\infty) = 0, \|\nabla \varphi\|_{\infty} \leq 1 \},
\]
where \( \text{Lip}(\mathbb{R}^d) \) and \( C^1(\mathbb{R}^d) \) denote, respectively, the set of real-valued Lipschitz functions (with
exponent 1) and continuously differentiable functions on \( \mathbb{R}^d \), and \( \Delta \varphi \) denotes the Laplacean
of \( \varphi \) in the sense of distributions. These capacities can be understood as a high-dimensional
version of the analytic and continuous analytic capacities \( \gamma \) and \( \alpha \), respectively.

There have been many advances in the last years concerning these capacities. For instance,
it has been proved that both \( \kappa \) and \( \kappa_c \) are countably semiadditive (see [Vo] and [RT1]),
among other geometric properties (see [DM] or [MP], for example). However, there are
still many open problems in this area. Let us stress that the mentioned results exhibit the
intimate relation between \( \kappa \) and \( \kappa_c \) with \( \gamma \) and \( \alpha \). For example, in the plane, given an
admissible function \( \varphi \) for \( \kappa(E) \), the distribution \( \partial \varphi \) coincides with an admissible function \( f \)
for \( \gamma \) (this is straightforward from the definitions, using that \( \partial f = -\partial \varphi = -\frac{1}{4}\Delta \varphi \) in the sense
of distributions), and since \( \langle 1, \Delta \varphi \rangle = f'(\infty) \) (modulo constants) by applying Stoke’s theorem
in a ball containing \( E \) (see [Pr, Lemma 2.2(9)])], one can easily show that \( \kappa \leq \gamma \) (modulo
constants). The analogous inequality holds for \( \kappa_c \) and \( \alpha \).
Concerning the analytic capacities, there exists a dual characterization of \( \alpha \) in terms of the so-called Garabedian function, which can be stated as follows: let \( F \subset \mathbb{C} \) be the closure of a bounded domain with smooth boundary. Then,
\[
\alpha(F) = \inf \left\{ \frac{1}{2\pi} \int_{\partial F} |h(z)| ds : h \in E^1(F^c), h(\infty) = 1 \right\},
\]
where \( ds \) denotes the arc length and \( E^1(F^c) \) is the Smirnov class of functions \( h \) analytic in \( F^c \cup \{\infty\} \) such that \( \int_{\partial F} |h(z)| ds < \infty \) (see [Dr, Chapter 10] for the precise definition and some properties of \( E^1 \)). Moreover, the infimum is attained, and the function \( \psi \) that solves the extremal problem is called the Garabedian function of \( F \); so \( \alpha(F) = \frac{1}{2\pi} \int_{\partial F} |\psi(z)| ds \). Let us mention that the classical proof of this characterization of \( \alpha \) has three basic ingredients: duality, the Hahn-Banach theorem, and the F. and M. Riesz theorem.

Keeping the analogy between \( \kappa_c \) and \( \alpha \) in mind, we will give a dual characterization of the \( C^1 \) harmonic capacity \( \kappa_c \) in the same spirit as the one with the Garabedian function for \( \alpha \). This characterization states that \( \kappa_c \) coincides with the solution of an extremal problem concerning some vector measures which annihilate a fixed space of harmonic gradients. More precisely, we will show the following (see Theorem 2.2.1):

**Theorem.** Let \( E \subset \mathbb{R}^d \) be the closure of a bounded open set with smooth boundary, let \( d\sigma_o \) denote the surface measure of the outer boundary of \( E \) (i.e., the boundary of the unbounded component of the complement of \( E \subset \mathbb{R}^d \)), and let \( \eta \) be the outer unit normal vector in the boundary of \( E \).

Then, \( \kappa_c(E) = \min \|\eta d\sigma_o + \mu\| \), where the minimum is taken over all finite real Borel vector measures \( \mu \) supported in \( E \) and such that \( \int \nabla \varphi d\mu = 0 \) for all \( \varphi \in C^1(\mathbb{R}^d) \) with \( \text{supp} \Delta \varphi \subset E \) and \( \nabla \varphi(\infty) = 0 \) (\( \|\eta d\sigma_o + \mu\| \) denotes the variation norm of \( \eta d\sigma_o + \mu \)). The minimum is attained at some vector measure which is called Garabedian measure of \( E \).

The proof of this characterization of \( \kappa_c \) also uses a duality argument and the Hahn-Banach theorem. However, there is not a higher-dimensional version of F. and M. Riesz theorem suitable for our setting, so our characterization of \( \kappa_c \) involves measures instead of continuous functions.

We will also give a measure-theoretic description and some properties of these vector measures, namely Theorem 2.2.2 and Proposition 2.2.3. In the end of the chapter, we will pose an open question about the Garabedian measure which we have not been able to answer (see Question 2.3.1), and we will give some consequences of a positive answer to that question (see Theorems 2.3.2, 2.3.3, and 2.3.4).

Similarly to what happens with \( \alpha \) and the Cauchy transform, one can define \( \kappa_c \) in terms of the Riesz transforms. One can show that, if \( \varphi \) is a defining function for \( \kappa \), then \( \varphi = \ldots \)
1/|x|^{d-2} \ast (\Delta \varphi) \text{ and } \nabla \varphi = x/|x|^d \ast (\Delta \varphi) \text{ for } d > 2, \text{ modulo constants. Hence, one can redefine the } \kappa_c \text{ capacity as } \kappa_c(E) = \sup(T,1), \text{ where now the supremum is taken over all distributions } T \text{ such that } \text{supp}(T) \subset E \text{ and } x/|x|^d \ast T \text{ coincides with a continuous vector function in } \mathbb{R}^d \text{ with norm bounded by 1 everywhere. Notice that } x/|x|^d \ast T \text{ is the Riesz transform of } T, \text{ in the sense of distributions. Thus, as before, this restatement of } \kappa_c \text{ connects the topic of this chapter with the Riesz transform.}

This chapter can be read independently.

**Chapters 3 and 4**

Variation and oscillation for singular integrals with odd kernel on Lipschitz graphs

The $\rho$-variation and oscillation for martingales and some families of operators have been studied in many recent papers on probability, ergodic theory, and harmonic analysis (see [Lé], [Bo], [JKRW], [CJRW1], [JSW], and [OSTTW], for example). Our purpose in these chapters is to establish some new results concerning the $\rho$-variation and oscillation for families of singular integral operators defined on Lipschitz graphs. More precisely, if $\mu$ denotes the $n$-dimensional Hausdorff measure on an $n$-dimensional Lipschitz graph in $\mathbb{R}^d$ ($0 < n < d$), the $\rho$-variation ($\rho > 2$) for the family of operators $T^\mu := \{T^\mu_\epsilon\}_{\epsilon > 0}$ given in (1) is defined by

\[
(V_\rho \circ T^\mu) f(x) := \sup_{\{\epsilon_m\}} \left( \sum_{m \in \mathbb{Z}} |T^\mu_{\epsilon_{m+1}} f(x) - T^\mu_{\epsilon_m} f(x)|^\rho \right)^{1/\rho} \quad \text{for } f \in L^1_{\text{loc}}(\mu) \text{ and } x \in \mathbb{R}^d,
\]

where the pointwise supremum is taken over all decreasing sequences $\{\epsilon_m\}_{m \in \mathbb{Z}} \subset (0,\infty)$. Given a decreasing sequence $\{r_m\}_{m \in \mathbb{Z}} \subset (0,\infty)$, the oscillation (with respect to $\{r_m\}$) for $T^\mu$ is defined by

\[
(O \circ T^\mu) f(x) := \sup_{\{\epsilon_m\}, \{\delta_m\}} \left( \sum_{m \in \mathbb{Z}} |T^\mu_{\epsilon_m} f(x) - T^\mu_{\delta_m} f(x)|^2 \right)^{1/2} \quad \text{for } f \in L^1_{\text{loc}}(\mu) \text{ and } x \in \mathbb{R}^d,
\]

where the pointwise supremum is taken over all sequences $\{\epsilon_m\}_{m \in \mathbb{Z}}$ and $\{\delta_m\}_{m \in \mathbb{Z}}$ such that $r_{m+1} \leq \epsilon_m \leq \delta_m \leq r_m$ for all $m \in \mathbb{Z}$. We are also interested in the $\rho$-variation and oscillation for the family $T^\mu_{\varphi} := \{T^\mu_{\varphi,\epsilon}\}_{\epsilon > 0}$, where

\[
T^\mu_{\varphi,\epsilon} f(x) := \int \varphi(|x-y|/\epsilon) K(x-y) f(y) \, d\mu(y), \quad x \in \mathbb{R}^d,
\]

and $\varphi : [0,\infty) \to \mathbb{R}$ is a non decreasing function of class $C^2$ such that $\chi_{[2,\infty)} \leq \varphi \leq \chi_{[1,\infty)}$ (the precise value of the constants is not important). We usually refer to $T^\mu_\epsilon$ and $T^\mu_{\varphi,\epsilon}$ as a rough and smooth truncation, respectively, of the singular integral with respect to the kernel
K and the measure \( \mu \). Our main result of Chapters 3 and 4 is summarized in the following theorem (see Main Theorem 3.0.1):

**Theorem.** Let \( 0 < n < d \) integers and \( \rho > 2 \). Let \( \mu \) be the \( n \)-dimensional Hausdorff measure restricted to an \( n \)-dimensional Lipschitz graph in \( \mathbb{R}^d \) with slope smaller than 1 (the slope of the graph is the Lipschitz constant of the function which defines it). Then, the operators \( \mathcal{V}_\rho \circ T^\mu \) and \( \mathcal{O} \circ T^\mu \) are bounded in \( L^p(\mu) \) for \( 1 < p < \infty \), from \( L^1(\mu) \) to \( L^{1,\infty}(\mu) \), and from \( L^{\infty}(\mu) \) to \( \text{BMO}(\mu) \).

The same holds without any restriction on the slope of the Lipschitz graph if one replaces \( \mathcal{V}_\rho \circ T^\mu \) and \( \mathcal{O} \circ T^\mu \) by \( \mathcal{V}_\rho \circ T^\rho \) and \( \mathcal{O} \circ T^\rho \).

This assumption on the slope of the Lipschitz graph is just a technical obstruction due to the methods we will use in this chapter, but, as we will see in Chapter 5, the \( \rho \)-variation and oscillation operators are actually bounded at least in \( L^2 \) for any Lipschitz graph, and even for more general measures.

As remarked in the beginning of the introduction, this results apply to the particular cases of the Cauchy transform (with \( d = 2 \), \( n = 1 \)) and the \( n \)-dimensional Riesz transforms on \( n \)-dimensional Lipschitz graphs in \( \mathbb{R}^d \). Concerning the applications, we will see that our results strengthen the celebrated result of R. Coifman, A. McIntosh, and Y. Meyer about the \( L^2 \) boundedness of the Cauchy transform on Lipschitz graphs. They also yield the boundedness of the so-called \( \lambda \)-jump operator for truncations of singular integrals and they give information on the speed of convergence of the principal value. Moreover, the \( \rho \)-variation and oscillation for the truncations of Riesz transforms are two operators intimately related with an important open problem posed by G. David and S. Semmes concerning boundedness of the maximal operator associated to Riesz transforms and uniform rectifiability. Since this will be the main topic of Chapter 5, we will not give more details here.

Concerning the background on the \( \rho \)-variation and oscillation, a fundamental result is Lépingle’s inequality [Lé], from which the \( L^p \) boundedness of the \( \rho \)-variation and oscillation for martingales follows, for \( \rho > 2 \) and \( 1 < p < \infty \). From this result on martingales, one deduces that the \( \rho \)-variation and oscillation are also bounded in \( L^p \) for averaging operators (also called differentiation operators, see [JKRW]). As far as we know, the first work dealing with the \( \rho \)-variation and oscillation for singular integral operators is the one of J. Campbell, R. L. Jones, K. Reinhold and M. Wierdl [CJRW1], where the \( L^p \) and weak \( L^1 \) boundedness of the \( \rho \)-variation (for \( \rho > 2 \)) and oscillation for the Hilbert transform was proved. Later on, there appeared other papers showing the \( L^p \) boundedness of the \( \rho \)-variation and oscillation for singular integrals in \( \mathbb{R}^d \) ([CJRW2]), with weights ([GT]), and for other operators such as the spherical averaging operator or singular integral operators on parabolas ([JSW]). Finally, we remark that, very recently, the case of the Carleson operator has been considered too ([LT],...
Chapter 5
Variation for Riesz transforms and uniform rectifiability

The relationship between \( L^2(\mu) \) boundedness of singular integrals and the geometric properties of a given measure \( \mu \) (such as rectifiability) is an area of research that has attracted much attention in the last years. There are influential contributions, for example, by G. David, P. Jones, P. Mattila, M. Melnikov, T. Murai, S. Semmes, X. Tolsa, J. Verdera, A. Volberg, etc. See [Pa], for example, for further names and references.

Let us now recall some definitions. A set \( E \subset \mathbb{R}^d \) is called \( n \)-rectifiable if there exists a countable family of \( n \)-dimensional \( C^1 \) submanifolds \( \{M_i\}_{i \in \mathbb{N}} \) such that \( E \setminus \bigcup_{i \in \mathbb{N}} M_i \) has vanishing \( n \)-dimensional Hausdorff measure. Similarly, a Borel measure \( \mu \) is said to be \( n \)-rectifiable if \( \mu(\mathbb{R}^d \setminus \bigcup_{i \in \mathbb{N}} M_i) = 0 \). A Borel measure \( \mu \) is said to be \( n \)-dimensional Ahlfors-David regular, or simply AD regular, if there exists some constant \( C \) such that \( C^{-1}r^n \leq \mu(B(x,r)) \leq Cr^n \) for all \( x \in \text{supp} \mu \) and \( 0 < r \leq \text{diam}(\text{supp} \mu) \). One also says that \( \mu \) is uniformly \( n \)-rectifiable if there exist \( \theta, M > 0 \) so that, for each \( x \in \text{supp} \mu \) and \( R > 0 \), there is a Lipschitz mapping \( g \) from the \( n \)-dimensional ball \( B^n(0,R) \subset \mathbb{R}^n \) into \( \mathbb{R}^d \) such that \( \text{Lip}(g) \leq M \) and \( \mu(B(x,R) \cap g(B^n(0,R))) \geq \theta R^n \), where \( \text{Lip}(g) \) stands for the Lipschitz constant of \( g \). Notice that uniform rectifiability implies rectifiability.

G. David and S. Semmes asked more than twenty years ago the still open question that follows (see, for example, [Pa, Chapter 7]):

**Question.** Is it true that an \( n \)-dimensional AD regular measure \( \mu \) is uniformly \( n \)-rectifiable if and only if \( R^n_\epsilon \) is bounded in \( L^2(\mu) \)?

Some comments are in order. In [DS1], G. David and S. Semmes proved the “only if” implication of the question above. Moreover, they gave a positive answer if one replaces, in the question, the \( L^2 \) boundedness of \( R^n_\epsilon \) by the \( L^2 \) boundedness of \( T_\epsilon^\mu \) for a wide class of odd kernels \( K \). In this direction, P. Mattila and D. Preiss proved in [MPr] the following result: let \( \mu \) be the \( n \)-dimensional Hausdorff measure restricted to a given \( n \)-dimensional AD regular closed set \( E \subset \mathbb{R}^d \). Assume that, for any \( C^\infty \) radial function \( h : \mathbb{R}^d \setminus \{0\} \rightarrow \mathbb{R} \) such that \( |h(x)| \leq C \) and \( |\nabla h(x)| \leq C|x|^{-1} \) for some fixed constant \( C > 0 \), the operators \( T_\epsilon^\mu \) defined by (1) with kernel \( K(x) = h(x)|x|^{-n-1}x \) are bounded in \( L^2(\mu) \) uniformly in \( \epsilon > 0 \). Then, \( E \) is \( n \)-rectifiable. The “if” implication in the question above was proved by P. Mattila, M. Melnikov and J. Verdera in [MMV] only for the case of the Cauchy transform, i.e. \( n = 1 \) and \( d = 2 \). Later on, G. David and J. C. Léger proved in [Le] that the \( L^2 \) boundedness \( C^n_\epsilon \)
implies that \( \mu \) is rectifiable, i.e., they obtained the corresponding “if” implication without the AD regularity assumption (for \( n = 1 \) and \( d = 2 \)).

When \( \mu \) is the \( n \)-dimensional Hausdorff measure on a set \( E \subset \mathbb{R}^d \) such that \( \mu(E) < \infty \), the rectifiability of \( \mu \) is also related to the existence of the principal value of the Riesz transform of \( \mu \), that is, the existence of \( R^\mu \frac{1}{x} \) for \( \mu \)-a.e. \( x \in \mathbb{R}^d \). For example, P. Mattila and M. Melnikov showed in [MM] that, if \( \mu \) is rectifiable, for all finite Borel measure \( \nu \) there exists \( R^\mu \frac{1}{x} \) for \( \mu \)-a.e. \( x \in \mathbb{R}^d \). In [MPr], P. Mattila and D. Preiss proved that, under the additional assumption that \( \liminf_{r \to 0} r^{-n} \mu(B(x,r)) > 0 \) for \( \mu \)-a.e. \( x \in E \), the rectifiability of \( E \) is equivalent to the existence of \( R^\mu \frac{1}{x} \) \( \mu \)-a.e. \( x \in E \). Later on, in [To10], X. Tolsa removed the assumption on the lower density of \( \mu \), i.e., he proved that a set \( E \) is rectifiable if and only if the principal value \( R^\mu \frac{1}{x} \) exists \( \mu \) almost everywhere. Let us mention that, for the case \( n = 1 \) and \( d = 2 \) (that is, for the Cauchy transform), the same results were obtained in [Ma2] (with some density assumptions) and in [To3] (by using the notion of curvature of measures). For other results dealing with principal values, Hausdorff measures, rectifiability, and related questions, see also [Hu], [MV], [Fa], [Vh], [RT2], and [To1], for example.

Using the results obtained in Chapter 3, we will prove the following theorem (see Corollary 5.0.10), which might be considered as a partial answer to the question above.

**Theorem.** Let \( \rho > 2 \). An \( n \)-dimensional AD regular measure \( \mu \) is uniformly \( n \)-rectifiable if and only if \( V_\rho \circ R^\mu \) is bounded in \( L^2(\mu) \), where \( R^\mu := \{R^\mu_\epsilon\}_{\epsilon > 0} \).

Therefore, \( V_\rho \circ R^\mu \) completely characterizes the \( n \)-AD regular measures \( \mu \) which are uniformly rectifiable. Notice that, as we will see in Chapter 3, the boundedness of \( V_\rho \circ R^\mu \) implies the existence of the principal values \( R^\mu \frac{1}{x} \), which in turn implies rectifiability. Thus our theorem yields stronger conclusions with, a priori, stronger hypotheses. The theorem above is the main motivation of this chapter and it is a direct consequence of Theorem 5.0.7, which is summarized in the following one:

**Theorem.** Let \( \mu \) be an \( n \)-dimensional AD regular Borel measure on \( \mathbb{R}^d \). Then, the following are equivalent:

(a) \( \mu \) is uniformly \( n \)-rectifiable,

(b) for any \( \rho > 2 \) and any \( T_\rho^\mu \) as in (1), the operator \( V_\rho \circ T^\mu \) is bounded in \( L^2(\mu) \),

(c) for some \( \rho > 0 \), the operator \( V_\rho \circ R^\mu \) is bounded in \( L^2(\mu) \).

The reader mostly interested in this theorem can skip Chapter 4 because, for its proof, we will only use the results proved in Chapter 3.

The second purpose of this chapter, which is summarized in the following theorem (see Theorem 5.0.12), is to improve the first endpoint estimate obtained in Chapters 3 and 4. Actually, the next theorem is an essential step to prove the previous one. We denote by \( M(\mathbb{R}^d) \) the space of finite complex Radon measures on \( \mathbb{R}^d \).
Theorem. Let \( \rho > 2 \) and let \( \mu \) be the \( n \)-dimensional Hausdorff measure restricted to an \( n \)-dimensional Lipschitz graph in \( \mathbb{R}^d \) with slope smaller than 1. Then, \( V_\rho \circ T \) and \( O \circ T \) are bounded operators from \( M(\mathbb{R}^d) \) to \( L_{1,\infty}^1(\mu) \) (see Definition 5.0.3 for the definition of \( T \)). In particular, \( V_\rho \circ T^\mu \) and \( O \circ T^\mu \) are of weak type \((1,1)\).

The same holds without the assumption on the slope of the Lipschitz graph if one replaces \( V_\rho \circ T \) and \( O \circ T \) by \( V_\rho \circ T^\varphi \) and \( O \circ T^\varphi \).

Let us finally mention that one can also pursue the results about the \( \lambda \)-jump operator mentioned in Chapters 3 and 4 in the setting of uniformly rectifiable measures.

Other comments

The chapters of this book contain the results that I have obtained, in a joint work with my PhD advisors Mark Melnikov and Xavier Tolsa, during my PhD research. These results have been settled in chronological order of appearance.

The most relevant parts of Chapter 1 are published in [M1]. Chapter 2 is a combination of the results published in [MMT1] and [MMT2]. The remaining part of the book has not been published yet, it has been separated and submitted in several papers (see [MT1] which has been accepted for publication, [MT2], and [M2]). They all are reprinted by permission of the corresponding publishers and co-authors.

Chapters 1 and 2 can be read independently of the rest of the book. Chapters 3, 4, and 5 should be read in order, but the reader mostly interested in the equivalence between uniform rectifiability and \( L^2 \) boundedness of the variation for Riesz transforms, which is a result obtained in Chapter 5, can go there directly from Chapter 3.

The original title of my PhD dissertation was Variation for Riesz transforms and analytic and Lipschitz harmonic capacities, defended in June 2011 at Universitat Autònoma de Barcelona. However, this book, which is the publication of my PhD dissertation in the form of a monograph, has a different title because of commercial reasons.
Chapter 1

Failure of rational approximation on some Cantor type sets

Consider a compact set $K$ of the complex plane. Let $A(K)$ be the algebra of continuous functions on $K$ which are analytic on the interior of $K$, and $R(K)$ the closure (with the uniform convergence on $K$) of the functions that are analytic on a neighborhood of $K$. Obviously, $R(K) \subseteq A(K)$.

In the 60’s, Vitushkin gave a description in terms of analytic capacity of the compact sets $K$ for which $R(K) = A(K)$ (see [Vt]), but there is still no characterization of those compact sets in a geometric way. Nevertheless, there have been important advances in this area recently, as can be seen in the articles of Xavier Tolsa [To7] and [To8] and the one of Guy David [Da2]. In this direction, Anthony G. O’Farrell raised the following question (private communication):

**Question 1.0.1.** Let $K_1$ and $K_2$ be two compact subsets of $[0,1]$ and define $K = (K_1 \times [0,1]) \cup ([0,1] \times K_2) \subseteq \mathbb{C}$. Is it true that $R(K) = A(K)$?

It is known that the identity holds if one of the compact sets $K_1$ or $K_2$ has no interior. For completeness, we include a proof of that fact at the end of the chapter. However, it was not known whether the identity holds or not in general. In this chapter we provide an example of a compact set $K$ which gives a negative answer to Question 1.0.1. The set $K$ is constructed as follows: Let $C(1/3)$ be the ternary Cantor set on the interval $[0,1]$, i.e.,

$$C(1/3) = \bigcap_{n=0}^{\infty} \bigcup_{j=1}^{2^n} I_{n}^{j},$$

where $I_{0}^{j} = [0,1]$ and each $I_{n}^{j}$ is an interval of length $3^{-n}$ obtained by dividing the intervals of length $3^{-n+1}$ in three equal parts and excluding the central part. Call $z_{n}^{j}$ the center of $I_{n}^{j}$.
Consider a sequence $\delta_n > 0$ such that $\delta_n < 3^{-n-1}$ and define $J^i_n = (z^i_n - \delta_n/2, z^i_n + \delta_n/2)$, where $z^i_n$ is the center of $I^i_n$. Let

$$E_m = [0, 1] \setminus \bigcup_{n=0}^{m} \bigcup_{j=1}^{2^n} J^j_n.$$ 

Finally, define $F_m = (E_m \times [0, 1]) \cup ([0, 1] \times E_m) \subseteq \mathbb{C}$ and put $K = \bigcap_{m=0}^{\infty} F_m$.

With this construction of $K$ we will prove the main result of the chapter:

**Theorem 1.0.2.** For a suitable choice of the sequence $\delta_n$, $R(K) \neq A(K)$.

In the whole chapter $\mathcal{M}^1$ stands for the *1-dimensional Hausdorff content*, $\gamma$ denotes the *analytic capacity*, and $\alpha$ denotes the *continuous analytic capacity* (see [Vt]). Remember that, given a compact set $F \subseteq \mathbb{C}$, $\gamma(F) := \sup |f'(\infty)|$, where the supremum is taken over all functions $f : \mathbb{C} \rightarrow \mathbb{C}$ which are analytic on $\mathbb{C} \setminus F$, and uniformly bounded by 1 on $\mathbb{C}$. If $f$ satisfies all these properties, we say that $f$ is *admissible* for $\gamma$ and $F$. The continuous analytic capacity $\alpha$ is defined as $\gamma$ but one also requires the defining functions to be continuous on $\mathbb{C}$.

Let us recall that the most relevant parts of this chapter are published in [M1].

![Figure 1.1](image.png) 

Figure 1.1: This is a picture of the compact set $F_2$. The rectangles inside $[0, 1]^2$ are the holes of $F_2$, and the bold lines on the sides of $[0, 1]^2$ correspond to the subset of the real line $\bigcup_{n=0}^{2^n} \bigcup_{j=1}^{2^n} J^j_n$. 


1.1 Proof of the main result

In the following two lemmas, we shall obtain some estimates of the Hausdorff content of \([0, 1]^2 \setminus K\) that will be useful in showing that the algebras \(R(K)\) and \(A(K)\) are not equal for a suitable choice of the sequence \(\delta_n\).

**Lemma 1.1.1.** Fix \(n_0 \in \mathbb{N}\) and \(\delta > 0\) such that \(\delta < 3^{-n_0+2}\). Define \(\tilde{J}_n^j = (z_n^j - \delta/2, z_n^j + \delta/2)\), \(R_n^j = \tilde{J}_n^j \times [0, \delta]\) and

\[
R = \bigcup_{n=0}^{n_0} \bigcup_{j=1}^{2^n} R_n^j.
\]

Then \(\mathcal{M}^1(R) < 8\delta^n\), where \(\eta = 1 - \frac{1}{\log_2 3} > 0\).

**Proof.** Since \(R\) is the union of the squares \(R_n^j\) for \(0 \leq n \leq n_0\) and \(1 \leq j \leq 2^n\) and each square has side length \(\delta\), we have

\[
\mathcal{M}^1(R) \leq \sum_{n=0}^{n_0} \sum_{j=1}^{2^n} \delta = \delta(2^{n_0+1} - 1) \leq \delta 2^{n_0+1}.
\]

The inequality \(\delta < 3^{-n_0+2}\) is equivalent to \(n_0 < 2 - \log_3 \delta\). Then, using the fact that \(\log_3 \delta = \log_2 \delta / \log_2 3\), we can deduce that

\[
\delta 2^{n_0+1} < \delta 2^{2 - \log_2 \delta} = \delta 2^{3 - \log_2 \delta} = 8\delta^{1 - \frac{1}{\log_2 3}} = 8\delta^n.
\]

\[\square\]

As we will see in the proof of the following lemma, the important fact of the preceding one is that \(\mathcal{M}^1(R)\) is bounded by something that tends to zero as \(\delta\) decreases rather than the exact value of the bound.

**Lemma 1.1.2.** For every \(\varepsilon > 0\) there exists a sequence \(\delta_n\) such that \(\mathcal{M}^1([0, 1]^2 \setminus K) < \varepsilon\).

**Proof.** Put \(G = [0, 1]^2 \setminus K\). Consider the crosses \(P_n^k\) for \(k = 1, \ldots, 4^n\) defined in the following way (see Figure 1.1 to understand the construction):

\[
P_0^1 = (J_0^1 \times [0, 1]) \cup ([0, 1] \times J_0^1),
\]

\[
P_1^1 = (J_1^1 \times [0, 1/3]) \cup ([0, 1/3] \times J_1^1), \quad P_1^2 = (J_1^2 \times [0, 1/3]) \cup ([2/3, 1] \times J_1^1),
\]

\[
P_1^3 = (J_1^3 \times [2/3, 1]) \cup ([0, 1/3] \times J_1^2), \quad P_1^4 = (J_1^2 \times [2/3, 1]) \cup ([2/3, 1] \times J_1^2),
\]

\[
P_2^1 = (J_2^1 \times [0, 1/9]) \cup ([0, 1/9] \times J_2^1), \quad P_2^2 = (J_2^2 \times [0, 1/9]) \cup ([2/9, 1/3] \times J_2^1),
\]

\[
P_2^3 = (J_2^3 \times [0, 1/9]) \cup ([2/3, 7/9] \times J_2^1), \quad P_2^4 = (J_2^4 \times [0, 1/9]) \cup ([8/9, 1] \times J_2^1),
\]

\[
P_2^5 = (J_2^5 \times [2/9, 1/3]) \cup ([0, 1/9] \times J_2^2), \ldots
\]
It is clear that \( G \subseteq \bigcup_{n=0}^{\infty} \bigcup_{k=1}^{4^n} P_n^k \). By construction, we also have \( M^1(P_n^1 \cap G) = M^1(P_n^k \cap G) \) for all \( k = 1, \ldots, 4^n \). Therefore,

\[
M^1(G) \leq \sum_{n=0}^{\infty} 4^n M^1(P_n^1 \cap G).
\]

Denote by \( X_n \) the horizontal strip of the cross \( P_n^1 \) and \( Y_n \) the vertical one. Because of the symmetry of the compact set \( K \) and the subadditivity of \( M^1 \), \( M^1(P_n^1 \cap G) \leq 2M^1(X_n \cap G) \).

Observe that \( G \) is a countable union of rectangles and on \( X_n \) all those rectangles have sides of length less than or equal to \( \delta_n \). So, the set \( 3^n(x, X_n \cap G) = \{3^n x : x \in X_n \cap G\} \) can be included by a translation in a set \( R := \bigcup_{n=0}^{n_0} \bigcup_{j=1}^{2^n} R_n^j \) like the one of the preceding lemma if we take \( \delta = 3^n \delta_n \) and \( n_0 \in \mathbb{N} \) such that \( 3^{-n_0+1} \leq \delta < 3^{-n_0+2} \). Applying the lemma we obtain, \( M^1(X_n \cap G) < 3^{-n} 8(3^n \delta_n)^n = 3^n(\eta-1)8\delta_n^n \) with \( \eta = 1 - \frac{1}{\log_3 8} \) and then,

\[
M^1(G) \leq 8 \sum_{n=0}^{\infty} 4^n M^1(X_n \cap G) < 8 \sum_{n=0}^{\infty} 4^n 3^n(\eta-1)\delta_n^n.
\]

Given \( \varepsilon > 0 \), it is easy to find a decreasing sequence \( \delta_n \) that makes the last sum less than \( \varepsilon \), because \( \eta > 0 \).

\[\square\]

**Proof of Theorem 1.0.2.** As Vitushkin proved in [Vt] (see also [Ga, Theorem VIII.8.2]), \( R(K) = A(K) \) if and only if \( \alpha(D \setminus K) = \alpha(D \setminus \text{int}K) \) for every bounded open set \( D \).

If \( C := C(1/3) \times C(1/3) \), we known that \( \alpha(C) > 0 \) because \( \text{dim}(C) > 1 \), where \( \text{dim}(\cdot) \) denotes the Hausdorff dimension. Observe that \( C \subseteq \partial K \) and it does not depend on the chosen sequence \( \delta_n \). This implies that \( \alpha(\partial K) \geq \alpha(C) \), so it is guaranteed a minimum of continuous analytic capacity on the boundary of \( K \) for any sequence \( \delta_n \).

Observe also that \( \alpha([0,1]^2 \setminus \text{int}K) = \alpha((0,1)^2 \setminus \text{int}K) \) because \( \partial([0,1]^2) \) is negligible (see [Ga, Chapter VIII]). Therefore,

\[
\alpha(C) \leq \alpha(\partial K) \leq \alpha([0,1]^2 \setminus \text{int}K) = \alpha((0,1)^2 \setminus \text{int}K).
\]

On the other hand, by the preceding lemma we can find a sequence \( \delta_n \) such that \( M^1([0,1]^2 \setminus K) \leq \alpha(C)/2 \). If we take \( \alpha \leq M^1 \) into account (see [Ga, Chapter VIII]), we can deduce that

\[
\alpha((0,1)^2 \setminus K) \leq M^1([0,1]^2 \setminus K) \leq \alpha(C)/2 < \alpha(C) \leq \alpha((0,1)^2 \setminus \text{int}K).
\]

These inequalities show that the necessary condition for \( R(K) = A(K) \) in Vitushkin’s theorem does not hold for \( D = (0,1)^2 \). So, for that sequence \( \delta_n \) we have \( R(K) \neq A(K) \).

\[\square\]
1.2 \( A(K) = R(K) \) when \( K_1 \) has no interior

We proceed to give an affirmative answer to Question 1.0.1 with the assumption that \( K_1 \) has no interior. We need an auxiliary lemma, namely Lemma 1.2.1, that we guess is already known, but we give the proof for completeness. We should say that the lemma states something much more general than what we need for our purposes, and we think that it has its own interest.

Lemma 1.2.1. Fix a line \( L \) and consider a disjoint family of squares \( \{Q_n\}_{n \in \mathbb{N}} \) such that all the squares have one side lying on \( L \) and the centers of the square are on the same half plane delimited by \( L \). Suppose also that \( 2 \text{int} Q_i \cap 2 \text{int} Q_j = \emptyset \) when \( i \neq j \) and \( \sum_n d_n < \infty \), where \( d_n \) is the side length of \( Q_n \). Let \( E_n \) be a subset of \( Q_n \). There exists an absolute constant \( C > 0 \) such that

\[
\sum_{n \in \mathbb{N}} \gamma(E_n) \leq C \gamma(\bigcup_{n \in \mathbb{N}} E_n).
\]

Proof. Given \( x, y, z \in \mathbb{C} \) and a Radon measure \( \mu \), let \( c(x, y, z) = R(x, y, z)^{-1} \), where \( R(x, y, z) \) is the radius of the circumference passing through \( x, y, z \) (with \( R(x, y, z) = \infty \), \( c(x, y, z) = 0 \) if \( x, y, z \) lie on a same line), and we set

\[
c^2_\mu(x) = \iint c(x, y, z)^2 \, d\mu(y) \, d\mu(z),
\]

and

\[
c^2(\mu) = \int c^2_\mu(x) \, d\mu(x) = \iiint c(x, y, z)^2 \, d\mu(x) \, d\mu(y) \, d\mu(z).
\]

The quantity \( c^2(\mu) \) is usually called Melnikov-Menger curvature of \( \mu \).

If we denote by \( M^+(F) \) the set of positive Radon measures \( \nu \) supported on \( F \), it is known (see [To9, Theorems 12 and 15], for example) that, for a given set \( F \),

\[
\gamma(F) \approx \sup \{ \mu(F) : \mu \in M^+(F), \, \mu(B(z, r)) \leq r \ \forall \ z \in \mathbb{C}, \ \forall \ r > 0, \ \text{and} \ c^2(\mu) \leq \mu(F) \}.
\]

Let \( \mu_n \) be a positive Radon measure realizing the analytic capacity of \( E_n \), i.e., such that \( \mu_n(E_n) \approx \gamma(E_n) \), \( \text{supp} \mu_n \subseteq E_n \), \( \mu_n(B(z, r)) \leq r \ \forall \ z \in \mathbb{C}, \ \forall \ r > 0 \), and \( c^2(\mu_n) \leq \mu_n(E_n) \). Define \( E := \bigcup_{n \in \mathbb{N}} E_n \) and consider the measure \( \mu = \sum_{n \in \mathbb{N}} \mu_n \). It is easy to see that \( \mu(B(z, r)) \leq Cr \) using the fact that the \( E_n \) are aligned. So, if we prove that \( c^2(\mu) \leq C \mu(E) \), we will have

\[
\sum_{n \in \mathbb{N}} \gamma(E_n) \approx \sum_{n \in \mathbb{N}} \mu_n(E_n) \leq \sum_{n \in \mathbb{N}} \mu_n(E) = \mu(E) \leq C \gamma(E),
\]

and we will be done. We decompose

\[
c^2(\mu) = \sum_{i,j,k \in \mathbb{N}} \iint c(x, y, z)^2 \, d\mu_i(x) \, d\mu_j(y) \, d\mu_k(z) =: \sum_{i,j,k \in \mathbb{N}} c^2(\mu_i, \mu_j, \mu_k) = I + II + III,
\]
where \( I \) corresponds to the sum over all the terms whose three indices coincide, \( II \) is the sum over the ones whose three indices are different and \( III \) corresponds to the rest of the terms. We will estimate each of these sums separately.

First of all, using that \( c^2(\mu_i, \mu_i, \mu_i) = c^2(\mu_i) \) one easily verifies that \( I \leq \mu(E) \). We deal now with the term \( III \). By elementary geometry, one can easily show that, for \( x, y, z \in \mathbb{C} \) pairwise different, \(|c(x, y, z)| = 2|\sin(yzx)||y - z|^{-1} \leq 2|y - z|^{-1} \), where \( yzx \) is the angle in the triangle of vertices \( x, y, z \) opposite to the side \( yz \). Given two numbers \( A \) and \( B \), we say that \( A \lesssim B \) if there exists an absolute constant \( C > 0 \) such that \( A \leq CB \), and we say that \( A \approx B \) if \( C^{-1}B \leq A \leq CB \). Notice that, because of \( 2Q_i \cap 2Q_j = \emptyset \) when \( i \neq j \), we have \(|y - z| \approx |y_i - z| \) for all \( y \in E_i \) and \( z \in E_j \), where \( y_i \) is the projection of the center of \( Q_i \) on \( L \). Therefore,

\[
III = 6 \sum_{i,j \neq i} c^2(\mu_i, \mu_i, \mu_j) = 6 \sum_{i,j \neq i} \int \int \int c^2(x, y, z) d\mu_j(z) d\mu_i(y) d\mu_i(x)
\]

\[
\lesssim \sum_{i,j \neq i} \int \int \int \frac{1}{|y-z|^2} d\mu_j(z) d\mu_i(y) d\mu_i(x) \lesssim \sum_i \mu_i(E_i) \int \int \int \frac{1}{|y_i-z|^2} d\mu_j(z) d\mu_i(y)
\]

\[
\lesssim \sum_i \mu_i(E_i) \int \int_{|y_i-z| \geq d_i/4} \frac{1}{|y_i-z|^2} d\mu(z) d\mu_i(y).
\]

Since \( \mu_i \) and \( \mu \) have linear growth, we have \( \int_{|y_i-z| \geq d_i/4} |y_i-z|^{-2} d\mu(z) \leq C/d_i \) and \( \mu_i(\mathbb{C}) = \mu(Q_i) \leq C d_i \). Thus,

\[
\sum_i \mu_i(E_i) \int \int_{|y_i-z| \geq d_i/4} \frac{1}{|y_i-z|^2} d\mu(z) d\mu_i(y) \lesssim \sum_i \mu_i(E_i) = \sum_i \mu_i(E) = \mu(E).
\]

Finally, we estimate \( II = 6 \sum_{i<j<k} \int \int \int c^2(x, y, z) d\mu_k(z) d\mu_j(y) d\mu_i(x) \). In [To2, Lemma 2.4] it is shown that given three pairwise different points \( x, y, z \in \mathbb{C} \) and \( x' \in \mathbb{C} \) such that \(|x - y| \approx |x' - y|\), then \(|c(x, y, z) - c(x', y, z)| \leq C|x - x'||x - y|^{-1}|x - z|^{-1} \). Let \( x \in Q_i, y \in Q_j, z \in Q_k \). If \( x_i, y_j, z_k \) denote the projections of the centers of \( Q_i, Q_j, Q_k \) on \( L \), respectively, \( c(x_i, y_j, z_k) = 0 \). Then,

\[
|c(x, y, z)| = |c(x, y, z) - c(x_i, y_j, z_k)|
\]

\[
\leq |c(x, y, z) - c(x_i, y, z)| + |c(x_i, y, z) - c(x_i, y_j, z)| + |c(x_i, y_j, z) - c(x_i, y_j, z_k)|
\]

\[
\lesssim \frac{|x - x_i|}{|x_i - y||x_i - z|} + \frac{|y - y_j|}{|x_i - y_j||y_j - z|} + \frac{|z - z_k|}{|x_i - z_k||y_j - z_k|}.
\]

Therefore,

\[
|c(x, y, z)|^2 \lesssim \left( \frac{|x - x_i|}{|x_i - y||x_i - z|} \right)^2 + \left( \frac{|y - y_j|}{|x_i - y_j||y_j - z|} \right)^2 + \left( \frac{|z - z_k|}{|x_i - z_k||y_j - z_k|} \right)^2. \tag{1.2.1}
\]
Observe that
\[ \sum_{i<j<k} \iiint_{} \left( \frac{|x-x_j|}{|x_i-| \, x_i|-z|} \right)^2 d\mu_k(z) \, d\mu_j(y) \, d\mu_i(x) \lesssim \sum_{i<j<k} \mu_i(E_i) \iint_{} \frac{d\mu_j(y) \, d\mu_k(z)}{|x_i-y|^2 |x_i-z|^2} \]
\[ \lesssim \sum_{i \in \mathbb{N}} \mu_i(E_i) \iint_{} \frac{d\mu_j(y) \, d\mu_k(z)}{|x_i-y|^2} \sum_{k>j} \int_{} \frac{d\mu_k(z)}{|x_i-z|^2} \]
\[ \leq \sum_{i \in \mathbb{N}} \mu_i(E_i) \iint_{} \frac{d\mu(y)}{|x_i-y|^2} \int_{} \frac{d\mu(z)}{|x_i-z|^2} \lesssim \sum_{i \in \mathbb{N}} \mu_i(E_i), \]
where we used the linear growth of \( \mu \) in the last inequality. As before, \( \sum_{i \in \mathbb{N}} \mu_i(E_i) \leq \mu(E) \).

With similar arguments, one can also show that
\[ \sum_{i<j<k} \iiint_{} \left( \frac{|y-y_j|}{|x_i-y_j| |y_j-z|} \right)^2 d\mu_k(z) \, d\mu_j(y) \, d\mu_i(x) \leq C \mu(E) \]
and
\[ \sum_{i<j<k} \iiint_{} \left( \frac{|z-z_k|}{|x_i-z_k| |y_j-z_k|} \right)^2 d\mu_k(z) \, d\mu_j(y) \, d\mu_i(x) \leq C \mu(E). \]

Now we can use these estimates together with (1.2.1) to obtain \( II \leq C \mu(E) \), which finishes the proof of the lemma.

\( \square \)

**Theorem 1.2.2.** Let \( K_1, K_2 \subseteq [0,1] \) be two compact sets and define \( K = (K_1 \times [0,1]) \cup ([0,1] \times K_2) \). Suppose that \( K_1 \) has no interior. Then, \( R(K) = A(K) \).

**Proof.** By Vitushkin’s theorem, it is known that \( R(K) = A(K) \) if and only if there exists an absolute constant \( C > 0 \) such that \( \alpha(Q \setminus \text{int}K) \leq C \alpha(Q \setminus K) \) for all open squares \( Q \).

Fix a square \( Q \) of side length \( l > 0 \). We can assume that \( Q \setminus K \) is not empty, so there exists a square \( F \subseteq Q \setminus K \). Let \( \pi_x \) and \( \pi_y \) be the projections onto the horizontal and vertical coordinate axes respectively. Then, \( \pi_y(F) \subseteq \pi_y(Q) \setminus K_2 \) and we can find an interval \( F_y \subseteq \pi_y(F) \) of length \( l/n \) for \( n \) big enough.

On the other hand, by taking \( n = 4m \) for some \( m \) big enough, if we split \( \pi_x(Q) \) into intervals \( I_j \) for \( j = 1, \ldots, n \) with pairwise disjoint interiors and length \( l/n \), we can also find intervals \( F^j_x \subseteq (\pi_x(Q) \setminus K_1) \cap I_{4j} \) for \( j = 1, \ldots, m \) (notice that \( 2I_{4r} \cap 2I_{4s} = \emptyset \) if \( r \neq s \)), because \( K_1 \) has no interior. Therefore, \( \bigcup_{j=1}^m (F^j_x \times F_y) \subseteq Q \setminus K \) and \( \gamma(F^j_x \times F_y) \geq C_0 l/n \).

Now we are ready to use Lemma 1.2.1 with the squares \( Q_j = I_{4j} \times F_y \) and the subsets \( E_j = F^j_x \times F_y \subseteq Q_j \) for \( j = 1, \ldots, m \) (recall that we have set \( n = 4m \)). We obtain
\[ \alpha(Q \setminus \text{int}K) \leq \alpha(Q) \approx l \lesssim \sum_{j=1}^m \gamma(F^j_x \times F_y) \lesssim \gamma\left( \bigcup_{j=1}^m (F^j_x \times F_y) \right) \leq \gamma(Q \setminus K) = \alpha(Q \setminus K), \]
where we also used in the last equality that \( \alpha \) and \( \gamma \) coincide on open sets. Since this holds for any open square \( Q \), then \( R(K) = A(K) \).

\( \square \)
We are grateful to Anthony O’Farrell for the communication of another proof of Theorem 1.2.2 which uses annihilating measures instead of Vitushkin’s theorem.
Chapter 2

A dual characterization of the $C^1$ harmonic capacity

Let $\text{Lip}(\mathbb{R}^d)$ be the set of real-valued Lipschitz functions (with exponent 1) on $\mathbb{R}^d$ and $C^1(\mathbb{R}^d)$ the set of real-valued continuously differentiable functions on $\mathbb{R}^d$. If $E \subset \mathbb{R}^d$ is a bounded set and

$$U'(E) = \{ \varphi \in \text{Lip}(\mathbb{R}^d) : \text{supp} \Delta \varphi \subset E, \nabla \varphi(\infty) = 0 \},$$

$$U'_c(E) = \{ \varphi \in C^1(\mathbb{R}^d) : \text{supp} \Delta \varphi \subset E, \nabla \varphi(\infty) = 0 \},$$

the Lipschitz and $C^1$ harmonic capacities of $E$ are defined by

$$\kappa(E) = \sup \{ \langle 1, \Delta \varphi \rangle : \varphi \in U'(E), \| \nabla \varphi \|_\infty \leq 1 \},$$

$$\kappa_c(E) = \sup \{ \langle 1, \Delta \varphi \rangle : \varphi \in U'_c(E), \| \nabla \varphi \|_\infty \leq 1 \},$$

where $\langle f, \Delta \varphi \rangle$ means the action of the compactly supported distribution $\Delta \varphi$ on a smooth function $f$, and $\| \nabla \varphi \|_\infty$ is the $L^\infty$ norm of the gradient $\nabla \varphi$ with respect to the Lebesgue measure $\mathcal{L}^d$ in $\mathbb{R}^d$. The symbol $\Delta$ denotes the Laplacean operator in $\mathbb{R}^d$.

In order to deal with the problem of harmonic approximation in the $C^1$-norm, P. Paramonov introduced in [Pr] the capacities $\kappa$ and $\kappa_c$, and gave a description, in terms of these capacities, of the compact sets $E \subset \mathbb{R}^d$ (with $d \geq 2$) such that any $C^1$ function harmonic in the interior of $E$ can be approximated in the $C^1$-norm by harmonic functions in a neighborhood of $E$.

The capacities $\kappa$ and $\kappa_c$ can be understood as high-dimensional versions of the so-called analytic and continuous analytic capacities $\gamma$ and $\alpha$ (respectively). Recall that, for a compact set $E \subset \mathbb{C}$,

$$\gamma(E) = \sup |f'(\infty)|,$$

where the supremum is taken over all analytic functions $f : \mathbb{C} \setminus E \to \mathbb{C}$ with $|f| \leq 1$ on $\mathbb{C} \setminus E$, and $f'(\infty) = \lim_{z \to \infty} z(f(z) - f(\infty))$. The continuous analytic capacity $\alpha$ has the
2. A dual characterization of the $C^1$ harmonic capacity

same definition as $\gamma$ except that one also requires the functions $f$ to be continuous in $\mathbb{C}$ and $|f| \leq 1$ everywhere.

The analytic capacity was first introduced by L. Ahlfors in [Ah] when he was characterizing the removable compact sets for bounded analytic functions in the plane. The continuous analytic capacity was defined by A. Vitushkin in [Vt] when he dealt with the problem of rational approximation in the uniform norm on compact sets of the plane. Both capacities have been studied by many authors since then (see [Gt] for a nice survey on results related with $\gamma$ and $\alpha$, and [Da2], [To7], or [To8] for more recent results). In particular, there exists a dual characterization of $\alpha$ that can be stated as follows: let $F \subset \mathbb{C}$ be the closure of a bounded domain with smooth boundary. Then,

$$\alpha(F) = \inf \left\{ \frac{1}{2\pi} \int_{\partial F} |h(z)|ds : h \in E^1(F^c), h(\infty) = 1 \right\},$$

where $ds$ denotes the arc length and $E^1(F^c)$ is the Smirnov class of functions $h$ analytic in $F^c \cup \{\infty\}$ such that $\int_{\partial F} |h(z)|ds < \infty$ (see [Dr, Chapter 10] for the definition and properties of $E^1$ in general domains). It is also proved that the infimum is attained, and the function $\psi$ that solves the extremal problem is called the Garabedian function of $F$; so $\alpha(F) = \frac{1}{2\pi} \int_{\partial F} |\psi(z)|ds$.

A classical way to construct the Garabedian function is to use the Hahn-Banach theorem and the F. and M. Riesz theorem. Observe that the quantity $\alpha(F)$ is the norm of the functional $f \mapsto f'(\infty) = \frac{1}{2\pi} \int_{\partial F} f(z)dz$ on the space of continuous functions outside $\text{int} F$ that are analytic outside $F$, which is a subspace of the continuous functions on $\partial F$. By the Hahn-Banach theorem one can find a measure $\mu$ supported on $\partial F$, orthogonal to the functions analytic outside $F$, and such that

$$\alpha(F) = \frac{1}{2\pi} \int_{\partial F} |dz + d\mu|.$$

The F. and M. Riesz theorem ensures that, in fact, $|dz + d\mu(z)| = |\psi(z)|ds$ for an analytic function $\psi$ that solves the extremal problem, where $ds$ denotes the arc length (see [Gt, Section I.4] for more details).

The aim of this chapter is to give a dual characterization of the $C^1$ harmonic capacity $\kappa_c$ in terms of some “Garabedian function”. This characterization is stated in Theorem 2.2.1 and it is based on the Hahn-Banach theorem, as can be done for the capacity $\alpha$.

Unfortunately, the F. and M. Riesz theorem can not be generalized to higher dimensions for our purposes, because the capacity $\kappa_c$ is defined in terms of gradients of harmonic functions and there are examples of measures orthogonal to those gradients which are not absolutely continuous with respect to the surface measure. One can simply take a unit tangent vector field $u$ on a closed curve $\Gamma$ of the boundary of the domain, and consider the measure $u \, ds|_\Gamma$. This means that we can not proceed exactly as in the classical case of the capacity $\alpha$ and,
instead of a “Garabedian function”, we will only have a “Garabedian measure” that minimizes some quantity analogous to the infimum in (2.0.2). We will adapt a theorem of B. Gustafsson and D. Khavinson about measures supported on the boundary of the domain and orthogonal to harmonic gradients (see Theorem 2.2.2) which, in particular, gives us a measure theoretic description of that minimal measure. We will also show in Proposition 2.2.3 a denseness property of the support of those measures.

Many properties of the Lipschitz and \( C^1 \) harmonic capacities were recently proven. The semiadditivity is a very important one, and it was obtained by A. Volberg in [Vo] for the capacity \( \kappa \). A little bit later, A. Ruiz de Villa and X. Tolsa proved in [RT1] that \( \kappa_c \) is also semiadditive. See [MP] and [Rz] for other interesting properties about these capacities.

In the last section of this chapter, we pose an open question that we have not been able to answer, namely Question 2.3.1. A positive answer gives some consequences on the capacity \( \kappa \), say Theorem 2.3.3 and Theorem 2.3.4. The first one states that \( \kappa(E) = \kappa(\partial_o E) \) for any compact set \( E \subset \mathbb{R}^d \), where \( \partial_o E \) denotes the outer boundary of \( E \) (i.e., the boundary of the unbounded component of \( E^c \)). This property is obvious for the capacity \( \gamma \) because of the defining conditions, but this is no longer trivial for \( \kappa \). Evidently, one has \( \kappa(E) \geq \kappa(\partial_o E) \). The difficulties appear when one tries to prove the reverse inequality. Observe that, by Green’s second theorem,

\[
\langle 1, \Delta \varphi \rangle = \int_{\partial_o V} \nabla \varphi \cdot \eta d\sigma \tag{2.0.3}
\]

for any \( \varphi \in U'(E) \), where \( V \) is a sufficiently regular neighborhood of \( E \), and \( \eta \) and \( d\sigma \) are the normal outward unit vector and surface measure of \( \partial V \), respectively. Suppose, for simplicity, that \( E \) is the closure of a bounded simply connected domain, so \( \partial_o E = \partial E \). One can try to prove that \( \kappa(E) \leq \kappa(\partial E) \) directly from the definition (2.0.1) and the identity (2.0.3). The idea would be to modify the functions \( \varphi \in U'(E) \) inside \( E \) to obtain functions \( \tilde{\varphi} \in U'(\partial E) \) such that \( \langle 1, \Delta \varphi \rangle = \langle 1, \Delta \tilde{\varphi} \rangle \). The problem is that one cannot ensure that the gradients \( \nabla \tilde{\varphi} \) are bounded by \( 1 \) in \( E \).

The second consequence of a positive answer to Question 2.3.1 is Theorem 2.3.4, which solves an open problem posed by A. Volberg (private communication). The problem can be stated as follows:

**Problem 2.0.3.** Let \( f \) be a real continuous function defined on the cube \( Q_0 = [0, \ell]^d \subset \mathbb{R}^d \) and let \( \Gamma = \{(x, f(x)) \in \mathbb{R}^d : x \in Q_0\} \) be the graph of \( f \). Prove that there exists a constant \( C > 0 \) depending only on \( d \) such that \( C \ell^d \leq \kappa(\Gamma) \).

Note that, if \( \text{diam}(\Gamma) \) is comparable to \( \ell \), Problem 2.0.3 states that \( \kappa(\Gamma) \geq C \text{diam}(\Gamma)^{n-1} \). This is a reasonable analogue of an important result in the area of analytic capacity which says that \( \gamma(E) \geq \frac{1}{4} \text{diam}(E) \) for any continuum (i.e., compact and connected set) \( E \subset \mathbb{C} \).
This classical result on $\gamma$ is a consequence of the $1/4$-theorem of Koebe (see [Ga, Theorem 2.1 of Chapter VIII]). A real variable proof of the inequality $\gamma(E) \geq C\text{diam}(E)$ for an absolute constant $C > 0$ was first obtained by P. Jones by using the notion of curvature of a measure (see [Pa, Section 3.5]).

One cannot expect this kind of estimates on $\kappa(E)$ for any continuum $E \subset \mathbb{R}^d$, because, for example, a segment in $\mathbb{R}^3$ has zero Lipschitz harmonic capacity. In fact, by using the identity (2.0.3), it is not difficult to show that $\kappa(E) = 0$ for any compact set $E \subset \mathbb{R}^d$ with zero $(d-1)$-Hausdorff measure. So, to obtain a reasonable analogue of the estimate of the analytic capacity of a continuum for the capacity $\kappa$, one has to restrict himself to continua with positive $(d-1)$-Hausdorff measure or, in an easier way, to graphs of continuous functions.

The structure of the chapter is the following. Section 2.1 is devoted to the preliminaries, where we will talk about vector measures, Lipschitz and $C^1$ harmonic capacities, and harmonicity at infinity (which includes the exterior Dirichlet and Neumann problems). With these notions, we will be ready to state and prove the dual characterization of $\kappa_c$ (i.e., Theorem 2.2.1), and the aforementioned description of the measures involved in the theorem (i.e. Theorem 2.2.2 and Proposition 2.2.3). This will be in Section 2.2. In Section 2.3 we state Question 2.3.1 and we prove the two announced properties of $\kappa$ that are given by an affirmative answer to the question.

Let us recall that this chapter is a combination of the results published in [MMT1], [MMT2], and [M2].

### 2.1 Preliminaries

In the whole chapter, we assume $d \geq 2$. The word smooth means of class $C^\infty$, wherever we talk about functions or the boundary of an open set. We write $\chi_E$ for the characteristic function of a set $E \subset \mathbb{R}^d$. The letter $C$ will denote a constant which may be different at different occurrences and which is independent of the relevant variables under consideration.

We denote by $\mathcal{C}(E)$ the set of real-valued continuous functions defined on a set $E \subset \mathbb{R}^d$, and by $\mathcal{C}(E)^d$ the cartesian product of $d$ spaces $\mathcal{C}(E)$.

Given a $C^\infty$ orientable manifold $M$ of dimension $n \leq d$ and $k \in \mathbb{N} \cup \{\infty\}$, let $\mathcal{C}^k(M)$ be the set of real-valued differentiable functions in $M$ such that their partial derivatives (with respect to the local coordinates chosen in $M$) of order less than $k+1$ exist and are continuous functions in $M$. In case when $\partial M \neq \emptyset$, we can take a system of local coordinates

$$\{U \subset M, y = (y_1, \ldots, y_n)\}$$

such that $U \cap M = y^{-1}(\{x_n \geq 0\})$ and $U \cap \partial M = y^{-1}(\{x_n = 0\})$, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. The partial derivative of a function $f : y(U) \rightarrow \mathbb{R}$ with respect to the coordinate $x_n$ at a
boundary point $p = (p_1, \ldots, p_{n-1}, 0) \in y(U \cap \partial M)$ is defined by the limit (if it exists)
\[
\left. \frac{\partial}{\partial x_n} f \right|_p = \lim_{t \to 0^+, t \to 0} \frac{f(p_1, \ldots, p_{n-1}, t) - f(p_1, \ldots, p_{n-1}, 0)}{t}.
\]

When $M \subset \mathbb{R}^d$, for any function $\varphi \in \mathcal{C}^1(M)$ we can identify the differential $d\varphi$ with a vector function in $\mathcal{C}(M)^d$, so we say that $\varphi \in \mathcal{C}^1(M)$ if $d\varphi \in \mathcal{C}(M)^d$. Clearly, $d\varphi = \nabla \varphi$ if $M$ is an open set. Notice that, if $M$ is the closure of an open set with smooth boundary, $\varphi|_{\partial M} \in \mathcal{C}^1(\partial M)$ for any $\varphi \in \mathcal{C}^1(M)$, and $(d\varphi)|_{\partial M}$ is the tangential part of $d\varphi$ with respect to $\partial M$.

It is an exercise to check that this definition of $\mathcal{C}^1(M)$ agrees with the classical one for closures of open sets with smooth boundary, i.e., $\mathcal{C}^1(M)$ is the set of continuous functions on $M$ such that their gradients on $\text{int} M$ can be extended to continuous vector functions on $M$. It also agrees with the definition of $\mathcal{C}^1(M)$ given in [Pr] or [Wh].

Our typical situation will be that $M$ is equal to $U$, $U$ or $\partial U$, for an open set $U \subset \mathbb{R}^d$ with smooth boundary.

### 2.1.1 Vector measures. The Riesz representation theorem

Given a Borel subset $E \subset \mathbb{R}^d$ and a vector function $f = (f_1, \ldots, f_n) : E \to \mathbb{R}^d$, define
\[
\|f\|_E = \sup\{|f(x)| : x \in E\}, \quad \text{where} \quad |f(x)| = \left(\sum_{i=1}^d (f_i(x))^2\right)^{1/2}.
\]
Clearly, $\mathcal{C}(E)^d$ with the norm $\| \cdot \|_E$ is a Banach space.

Given a bounded linear functional $\Lambda$ on $\mathcal{C}(E)^d$ and a subspace $\mathcal{F} \subset \mathcal{C}(E)^d$, define
\[
\|\Lambda\|_{\mathcal{F}} = \sup\{|\Lambda(f)| : f \in \mathcal{F}, \|f\|_E \leq 1\}.
\]
For simplicity, we write $\|f\|$ and $\|\Lambda\|$ instead of $\|f\|_E$ and $\|\Lambda\|_{\mathcal{C}(E)^d}$, respectively (when there is no confusion on what is $E$).

Let $\mathcal{M}(E)$ be the space of finite real Borel measures supported on $E$ and $\mathcal{M}(E)^d$ the cartesian product of $d$ spaces $\mathcal{M}(E)$. The elements of $\mathcal{M}(E)^d$ are commonly called vector measures. For $\mu = (\mu_1, \ldots, \mu_d) \in \mathcal{M}(E)^d$, define the variation of $\mu$ on a subset $F \subset E$ as
\[
|\mu|(F) = \sup\left\{ \sum_{j=1}^m |\mu(F_j)| : F = \biguplus_{j=1}^m F_j, \quad F_j \text{ is } \mu_i\text{-measurable } \forall i, j \right\},
\]
where $|\mu(F_j)|^2 = \sum_{i=1}^d (\mu_i(F_j))^2$ and $\biguplus_{j=1}^m F_j$ denotes a disjoint partition of $F$. Finally, define the total variation of $\mu$ as $\|\mu\|_E = |\mu|(E)$. It is proved that $|\mu|$ is a positive and finite measure on $E$ (see for example [La2, Theorem 3.1 of Chapter VII]). It is easily seen that $\| \cdot \|_E$ is a norm on the space $\mathcal{M}(E)^d$. 
Any vector measure $\mu \in \mathcal{M}(E)^d$ can be considered as a bounded linear functional $\langle \cdot, \mu \rangle : \mathcal{C}(E)^d \to \mathbb{R}$ by putting
$$\langle f, \mu \rangle = \int f \, d\mu = \sum_{i=1}^d \int f_i \, d\mu_i.$$  

On the other hand, the Riesz representation theorem (for scalar measures) shows that any bounded linear functional on $\mathcal{C}(E)^d$ can be represented as $\langle \cdot, \mu \rangle$ for some vector measure $\mu \in \mathcal{M}(E)^d$.

The concept of vector measure is treated in detail in some text books (see for example [DSz] or [DU]). However, usually in the literature, the Riesz representation theorem for vector measures is stated in a slightly different setting. The following result is the precise version of the Riesz representation theorem that we need.

**Theorem 2.1.1** (Riesz’ representation theorem). The map $\mu \mapsto \langle \cdot, \mu \rangle$ is an isometric isomorphism of $\mathcal{M}(E)^d$ onto the space of bounded linear functionals on $\mathcal{C}(E)^d$, so $\|\mu\|_E = \|\langle \cdot, \mu \rangle\|$ for all $\mu \in \mathcal{M}(E)^d$.

**Proof.** By the previous comments, it is enough to prove that the map $\mu \mapsto \langle \cdot, \mu \rangle$ is isometric to obtain the isomorphism. We have to check that for any $\mu \in \mathcal{M}(E)^d$,
$$\|\mu\|_E = \sup\{|\langle f, \mu \rangle| : f \in \mathcal{C}(E)^d, \|f\|_E \leq 1\}.$$  

We will see first that $\sup\{|\langle f, \mu \rangle| : f \in \mathcal{C}(E)^d, \|f\|_E \leq 1\} \leq \|\mu\|_E$. By density, it is enough to prove that $|\langle f, \mu \rangle| \leq \|\mu\|_E$ for simple vector functions of the form
$$f = \left(\sum_m a_m^1 \chi_{F_m}, \ldots, \sum_m a_m^d \chi_{F_m}\right),$$
where the sums are finite, the $a_m^i$’s are real numbers, the $F_j$’s are disjoint subsets of $E$, and $\|f\|_E = \sup_m (\sum_{i=1}^d (a_m^i)^2)^{1/2} \leq 1$. By the Cauchy-Schwartz inequality,
$$|\langle f, \mu \rangle| = \left| \sum_{i=1}^d \sum_m a_m^i \mu_i(F_m) \right| \leq \sum_m \left( \sum_{i=1}^d (a_m^i)^2 \sum_j (\mu_j(F_m))^2 \right)^{1/2} \leq \sup_m \left( \sum_{i=1}^d (a_m^i)^2 \right)^{1/2} \sum_m |\mu(F_m)| \leq \|f\|_E \|\mu\|_E \leq \|\mu\|_E.$$  

Let us prove now that $\|\mu\|_E \leq \sup\{|\langle f, \mu \rangle| : f \in \mathcal{C}(E)^d, \|f\|_E \leq 1\}$. Let $\nu$ be a positive measure such that $\mu_i$ is absolutely continuous with respect to $\nu$ for all $i = 1, \ldots, d$ (for example $\nu = \sum_{i=1}^d |\mu_i|$, where $|\mu_i|$ is the classical variation of the real measure $\mu_i$). Then $\mu_i = h_i \nu$, where $h_i$ is a $\nu$-measurable function. Observe that, if we put $h = (h_1, \ldots, h_d)$, then $\mu = h \nu$ and
$$\|\langle \cdot, \mu \rangle\| = \|\langle \cdot, h \nu \rangle\| : f \in \mathcal{C}(E)^d, \|f\|_E \leq 1 \leq \int_E |h| \, d\nu. \quad (2.1.1)$$
Consider the $\nu$-measurable vector function $g$ defined by $g(x) = h(x)/|h(x)|$ whenever $h(x) \neq 0$ and $g(x) = 0$ otherwise. Lusin’s theorem can be adapted to our situation to prove that given $\varepsilon > 0$ there exists $f_\varepsilon \in C(E)^d$ with $\|f_\varepsilon\|_E \leq \|g\|_E \leq 1$ and such that

$$\left| \int_E (g - f_\varepsilon) d\mu \right| < \varepsilon.$$  

This implies that

$$\int_E |h| d\nu = \int_E (g \cdot h) d\nu = \int_E g d\mu \leq \left| \int_E (g - f_\varepsilon) d\mu \right| + \int_E f_\varepsilon d\mu \leq \varepsilon + \left\| \langle \cdot, \mu \rangle \right\|$$

for all $\varepsilon > 0$. This estimate together with (2.1.1) proves that $\|\mu\|_E \leq \left\| \langle \cdot, \mu \rangle \right\|$ is enough to check that $|\mu(F)| \leq \int_F |h| d\nu$ for all $F \subset E \nu$-measurable. By a discrete version of Minkowski’s integral inequality,

$$|\mu(F)| = \left( \sum_{i=1}^d \left( \int_F d\mu_i \right)^2 \right)^{1/2} = \left( \sum_{i=1}^d \left( \int_F h_i d\nu \right)^2 \right)^{1/2} \leq \int_F \left( \sum_{i=1}^d h_i^2 \right)^{1/2} d\nu.$$

Therefore, $|\mu(F)| \leq \int_F |h| d\nu$, and the theorem is proved.

### 2.1.2 The Lipschitz and $C^1$ harmonic capacities

The fundamental solution $\phi_d$ for the Laplace equation $\Delta f = 0$ in $\mathbb{R}^d$ is defined by

$$\phi_d(x) = \begin{cases} a_d|x|^{2-d} & \text{if } d > 2, \\ a_d \log |x| & \text{if } d = 2, \end{cases}$$

where $a_d$ is a constant which depends on the dimension $d$.

We stated the definitions of the Lipschitz and $C^1$ harmonic capacities $\kappa$ and $\kappa_c$ in (2.0.1). The defining conditions for $U'(E)$ and $U'_c(E)$ imply that the functions $\varphi \in U'(E)$ are harmonic in $E^c$ and take the form $\varphi = \phi_d \ast \Delta \varphi + \text{constant}$, where this last equality is in the sense of distributions; but by the definitions of $\kappa$ and $\kappa_c$ we can suppose that, in fact, $\varphi = \phi_d \ast \Delta \varphi$. Recall that, if $T$ is a compactly supported distribution, then for each $\psi \in C^\infty(\mathbb{R}^d)$ with compact support, by definition (in view of parity of $\phi_d$),

$$\langle \phi_d \ast T, \psi \rangle = \langle T, \phi_d \ast \psi \rangle,$$

where $\phi_d \ast \psi(x) = \int \phi_d(y) \psi(x - y) dm(y)$ and $m$ is the Lebesgue measure on $\mathbb{R}^d$.

Therefore, if we take

$$U(E) = \{ \varphi \in \text{Lip}(\mathbb{R}^d) : \text{supp}\Delta \varphi \subset E, \varphi = \phi_d \ast \Delta \varphi \},$$

$$U_c(E) = \{ \varphi \in C^1(\mathbb{R}^d) : \text{supp}\Delta \varphi \subset E, \varphi = \phi_d \ast \Delta \varphi \},$$
we can redefine the Lipschitz and $C^1$ harmonic capacities by

$$
\kappa(E) = \sup \{ (1, \Delta \varphi) : \varphi \in U(E), \| \nabla \varphi \|_\infty \leq 1 \},
$$

$$
\kappa_c(E) = \sup \{ (1, \Delta \varphi) : \varphi \in U_c(E), \| \nabla \varphi \|_\infty \leq 1 \}.
$$

### 2.1.3 Harmonicity outside a compact set and at infinity.

Most of the material in this section can be found in [Fo].

**Definition 2.1.2.** For any set $E \subset \mathbb{R}^d \cup \{ \infty \}$ we define $E^* = \{ x/|x|^2 : x \in E \} \subset \mathbb{R}^d \cup \{ \infty \}$.

Given a function $u$ defined on a set $E \subset \mathbb{R}^d \setminus \{0\}$, define the Kelvin transform of $u$ by

$$
K_u(x) = |x|^{2-d} u(x/|x|^2), \quad \text{for } x \in E^*.
$$

**Theorem 2.1.3.** The Kelvin transform is its own inverse. If $V \subset \mathbb{R}^d \setminus \{0\}$ is an open set, then a function $u$ is harmonic in $V$ if and only if $K_u$ is harmonic in $V^*$.

**Definition 2.1.4.** If $E \subset \mathbb{R}^d$ is compact and $u$ is harmonic in $E^c$, then $u$ is harmonic at $\infty$ provided $K_u$ has a removable singularity at the origin.

**Theorem 2.1.5.** Suppose that $u$ is harmonic in $E^c$, where $E \subset \mathbb{R}^d$ is compact. Then, the following three conditions are equivalent:

1. $u$ is harmonic at $\infty$.
2. $|u(x)| = o(1)$ as $x \to \infty$ $(d > 2)$, or $|u(x)| = o(\log |x|)$ as $x \to \infty$ $(d = 2)$.
3. $|u(x)| = O(|x|^{2-d})$ as $x \to \infty$.

In particular, any function which is harmonic at infinity vanishes at infinity when $d > 2$ and is bounded when $d = 2$.

**Theorem 2.1.6** (Exterior Dirichlet problem). Let $F \subset \mathbb{R}^d$ be the closure of a bounded domain with smooth boundary. Given $h \in C(\partial F)$ there exists a unique function $u \in C(\overline{F^c})$ such that $u$ is harmonic in $F^c \cup \{ \infty \}$ and $u|_{\partial F} = h$. If $h \in C^\infty(\partial F)$, then $u \in C^\infty(\overline{F^c})$.

The last statement of the previous theorem is the elliptic regularity theorem (see [Fo, Proposition 7.37 and Theorem 6.33]).

**Theorem 2.1.7** (Exterior Neumann problem). Let $F \subset \mathbb{R}^d$ be the closure of a bounded domain with smooth boundary and $\eta$ the outward unit normal vector on $\partial F$. Let $V_1, \ldots, V_m$ be the bounded connected components of $F^c$ and $V_0$ the unbounded one. Let $h \in C(\partial F)$ such that

$$
\int_{\partial V_i} h \, d\sigma = 0 \quad \text{for all } i = 1, \ldots, m.
$$
2.1. Preliminaries

(1) Assume $d > 2$. Then, there exists a function $u \in C^1(\overline{F^c})$ such that $u$ is harmonic in $F^c \cup \{\infty\}$ and $(\nabla u \cdot \eta)|_{\partial F} = h$. The function $u$ is unique modulo functions which are constant on each bounded connected component of $F^c$.

(2) Assume $d = 2$. Then, there exists a function $u \in C^1(\overline{F^c})$ such that $u$ is harmonic in $F^c \cup \{\infty\}$ and $(\nabla u \cdot \eta)|_{\partial F} = h$ if and only if

$$\int_{\partial V_0} h d\sigma = 0.$$  

In that case, the function $u$ is unique modulo functions which are constant on each connected component of $F^c$.

In both cases, $u \in C^\infty(\overline{F^c})$ if $h \in C^\infty(\partial F)$.

Remark 2.1.8. Theorem 2.1.7 corresponds to [Fo, Theorem 3.40]. If we do not have the assumption $\int_{\partial V_0} h d\sigma = 0$ in Theorem 2.1.7(2), we can still find a function $u \in C^1(\overline{F^c})$ harmonic in $F^c$ and such that $(\nabla u \cdot \eta)|_{\partial F} = h$, by looking carefully at the proof of [Fo, Theorem 3.40]. Moreover, $u$ can be taken as

$$u(x) = \int_{\partial V_0} \log |x - y| u_0(y) d\sigma(y)$$  \hspace{1cm} (2.1.2)$$

for all $x \in V_0$ and for some $u_0 \in C(\partial V_0)$ depending on $h$. But now, $u$ may not be harmonic at infinity (because it may not be bounded) and we cannot ensure uniqueness in $F^c$ modulo constant functions. In fact, in [Fo, Lemma 3.31] it is shown that our particular solution $u$ is harmonic at infinity if and only if $\int_{\partial V_0} h d\sigma = 0$.

Theorem 2.1.9 (Green’s first theorem). Let $F \subset \mathbb{R}^d$ be the closure of a bounded domain with smooth boundary and $\eta$ the outward unit normal vector on $\partial F$. Let $u$ and $v$ be harmonic functions in $F^c$, $C^1$ up to $\partial F$, and such that

$$|(u(x)\nabla v(x) - v(x)\nabla u(x)) \cdot x| = o(|x|^{2-d})$$

when $|x| \to \infty$. Then,

$$\int_{\partial F} (\nabla u \cdot \eta) v d\sigma = \int_{\partial F} u (\nabla v \cdot \eta) d\sigma.$$ 

Proof. Let $B_R$ be the ball centered at the origin with radius $R$, and take $R > M$ such that $F \subset B_{R/2}$. Define $F_R = B_R \setminus F$ and let $\eta$ denote also the inward unit normal vector on $\partial B_R$.

By Green’s first theorem on $F_R$,

$$\int_{\partial F} (\nabla u \cdot \eta) v d\sigma = \int_{\partial F_R} (\nabla u \cdot \eta) v d\sigma - \int_{\partial B_R} (\nabla u \cdot \eta) v d\sigma$$

$$= \int_{\partial F_R} u (\nabla v \cdot \eta) d\sigma - \int_{\partial B_R} (\nabla u \cdot \eta) v d\sigma$$

$$= \int_{\partial F} u (\nabla v \cdot \eta) d\sigma + \int_{\partial B_R} (u (\nabla v \cdot \eta) - (\nabla u \cdot \eta) v) d\sigma.$$
For any $R$ big enough, by the assumption on $u$ and $v$, 
\[
\left| \int_{\partial B_R} (u(\nabla v \cdot \eta) - (\nabla u \cdot \eta)v) \, d\sigma \right| \leq \int_{\partial B_R} |u \nabla v \cdot \eta - v \nabla u \cdot \eta| \, d\sigma \leq o(R^{1-d})R^{d-1},
\]
and letting $R \to \infty$ we obtain the desired result. \hfill \Box

**Remark 2.1.10.** Let $F \subset \mathbb{R}^d$ be the closure of a bounded open set with smooth boundary. In the next section, we will need to apply Green’s first theorem to pairs of functions $\varphi$ and $u$, where $\varphi \in U_c(F)$ and $u$ is a function harmonic in $F^c \cup \{\infty\}$ and continuous up to $\partial F$. For this reason, we give now some estimates on the behavior of $\varphi$ and $u$ near infinity.

Let $\varphi \in U_c(F)$. By applying Green’s second theorem, we have that for all $x \in F^c$,
\[
\varphi(x) = \int \phi_d(x-y) \Delta_y \varphi(y) \, dm(y)
= \int_F (\text{div}_y(\phi_d(x-y) \nabla_y \varphi(y)) - \nabla_y \phi_d(x-y) \cdot \nabla_y \varphi(y)) \, dm(y)
= \int_{\partial F} \phi_d(x-y) \nabla_y \varphi(y) \cdot \eta(y) \, d\sigma(y) + \int_F \nabla \phi_d(x-y) \cdot \nabla_y \varphi(y) \, dm(y),
\]
where $\eta$ and $d\sigma$ are the outward unit normal vector and surface measure related to $\partial F$. Notice that, to apply Green’s second theorem, one should have enough regularity on $\varphi$. This could be assumed by convolving $\varphi$ with a suitable smooth approximation of the identity and taking limits.

Assume $d > 2$. By computing the derivatives of $\phi_d$, it is an exercise to see that any $\varphi \in U_c(F)$ satisfies the statement (3) of Theorem 2.1.5, so it is harmonic at infinity. It is proved in [Fo, Proposition 2.75] that any function $u$ harmonic outside $F$ and at infinity satisfies $|\nabla u(x) \cdot x| = O(|x|^{-d})$, so Theorem 2.1.9 can be applied to the pair $\varphi$ and $u$, and Green’s first theorem holds in that case.

The case $d = 2$ is a little bit different, because we cannot ensure that a function $\varphi \in U_c(F)$ has the required decay at infinity. We have the estimates $|\varphi(x)| = O(\log |x|)$ and $|\nabla \varphi(x)| = O(|x|^{-1})$ near infinity (and we cannot apply Theorem 2.1.5(2)). For a function $u$ harmonic outside $F$ and at infinity, we still have the estimates $|u(x)| = O(1)$ and $|\nabla u(x) \cdot x| = O(|x|^{-1})$, as can be seen in [Fo, Proposition 2.75]. These estimates are not enough to use Theorem 2.1.9, because $|u(x) \nabla \varphi(x) \cdot x| = O(1)$. But, if $u(\infty) = 0$, then $|u(x)| = o(1)$ and we can still apply Theorem 2.1.9 in that particular case.

### 2.2 The heart of the matter

In the whole section, $F \subset \mathbb{R}^d$ will be the closure of a bounded open set with smooth boundary and $\eta$ and $d\sigma$ will denote the outward unit normal vector and the surface measure of $\partial F$, respectively.
2.2. The heart of the matter

Consider the following normed spaces related to the compact set $F$:

\[
B(F) = \{f \in C(F)^d : f = \nabla \varphi, \varphi \in U_c(F)\}, \\
B(F)^\perp = \{\mu \in M(F)^d : \langle f, \mu \rangle = 0 \text{ for all } f \in B(F)\}, \\
bB(F)^\perp = \{\mu \in B(F)^\perp : \text{supp}\mu \subset \partial F\},
\]

where $B(F)$ is equipped with the norm $\|\cdot\|_F$ and the orthogonal spaces $B(F)^\perp$ and $bB(F)^\perp$ are equipped with the induced norm from $M(F)^d$. It is easily seen that the norm in $bB(F)^\perp$ induced by the space $M(F)^d$ coincides with the norm induced by the space $M(\partial F)^d$.

**Theorem 2.2.1.** Let $F \subset \mathbb{R}^d$ be the closure of a bounded open set with smooth boundary. Then,

\[
\kappa_c(F) = \min \left\{ \|\eta d\sigma_o + \mu\|_F : \mu \in B(F)^\perp \right\},
\]

where $d\sigma_o$ is the surface measure of $\partial_o F$, i.e., the restriction of $d\sigma$ to $\partial_o F$. The measure $\eta d\sigma_o + \mu$ with $\mu \in B(F)^\perp$ which attains the minimum is called the Garabedian measure of $F$ (it may not be unique).

**Proof.** By definition,

\[
\kappa_c(F) = \sup \{\langle 1, \Delta \varphi \rangle : \varphi \in U_c(F), \|\nabla \varphi\| \leq 1\}. \tag{2.2.1}
\]

Using Green’s second theorem, for each $\varphi \in U_c(F)$ we have

\[
\langle 1, \Delta \varphi \rangle = \int_{BF} \nabla \varphi \cdot \eta d\sigma = \int_{\partial_o F} \nabla \varphi \cdot \eta d\sigma + \int_{\partial F \setminus \partial_o F} \nabla \varphi \cdot \eta d\sigma = \int_{\partial_o F} \nabla \varphi \cdot \eta d\sigma_o,
\]

because each connected component of $\partial F \setminus \partial_o F$ is the boundary of a smooth bounded domain where $\varphi$ is harmonic.

Define the functional $\Phi : C(F)^d \to \mathbb{R}$ as $\Phi(f) = \int_{BF} f \cdot \eta d\sigma_o$. Then (2.2.1) becomes

\[
\kappa_c(F) = \sup \{\|\Phi(\nabla \varphi)\| : \nabla \varphi \in B(F), \|\nabla \varphi\| \leq 1\} = \sup \{\|\Phi(\nabla \varphi)\| : \nabla \varphi \in B(F), \|\nabla \varphi\|_F \leq 1\} = \|\Phi\|_{B(F)},
\]

because $|\nabla \varphi|$ is subharmonic when $\varphi$ is harmonic, and the maximum principle can be applied.

Clearly, a functional $\Phi$ on $C(F)^d$ is an extension of $\Phi|_{B(F)}$ if and only if $\Psi := \Phi - \Phi$ is orthogonal to $B(F)$, and for all such extensions $\|\Phi\|_{B(F)} \leq \|\Phi\|$. Hence,

\[
\|\Phi\|_{B(F)} = \min \left\{ \|\Phi\| : \Phi \text{ is an extension of } \Phi|_{B(F)} \to C(F)^d \right\} = \min \left\{ \|\Phi + \Psi\| : \Psi \text{ is orthogonal to } B(F) \right\} = \min \left\{ \|\eta d\sigma_o + \mu\|_F : \mu \in B(F)^\perp \right\},
\]

where we have used the Hahn-Banach theorem in the first equality, and Riesz’ representation theorem 2.1.1 in the third one. \qed
Let $A(F)$ be the set of smooth vector fields on $\partial F$. For any $g \in A(F)$, let $g_\tau$ be the tangential component of $g$ and $g_\eta$ the normal one, i.e., $g_\eta = g \cdot \eta$ and $g_\tau = g - g_\eta$. Notice that $g_\eta$ is a scalar function while $g_\tau$ is a vector one. Denote by $u_\eta$ the unique harmonic extension of $g_\eta$ to $F^c \cup \{\infty\}$ given by Theorem 2.1.6. Define $A_0(F) = \{g \in A(F) : u_\eta(\infty) = 0\}$. By Theorem 2.1.5, $A_0(F) = A(F)$ for $d > 2$.

For any $g \in A(F)$, the divergence of $g_\tau$ on $\partial F$ can be defined by its action on smooth and compactly supported functions $\varphi$ as

$$\int_{\partial F} (\text{div} g_\tau) \varphi d\sigma = -\int_{\partial F} g_\tau \cdot \nabla \varphi d\sigma.$$ 

In the right hand side, we can replace $\nabla \varphi$ by $(\nabla \varphi)_\tau$, which only depends on the values of $\varphi$ on $\partial F$ (remember that $(\nabla \varphi)_\tau = d(\varphi|_{\partial F})$). This definition agrees with the classical one of divergence in the context of Riemannian manifolds in $\mathbb{R}^d$ (see [Wa]).

Let $G(F)$ be the set of measures $vds|_\Gamma$, where $\Gamma$ is a piecewise $C^1$ closed curve in $F$, $ds|_\Gamma$ is the arc length measure on $\Gamma$ and $v$ is a continuous tangential vector field on $\Gamma$ with constant modulus.

The following theorem gives a description of the measures in $B(F)^\perp$ or $bB(F)^\perp$, and, in particular, of the Garabedian measure of $F$. It is a minor modification of [GK, Theorem 3.1], where the annihilators of the space of harmonic gradients inside $F$ are studied. In our case, we need the harmonicity outside $F$. Our proof is almost the same as the one of [GK, Theorem 3.1] and, for completeness, we include all the detailed arguments.

**Theorem 2.2.2.** Let $g \in A_0(F)$. Then $gd\sigma \in bB(F)^\perp$ if and only if

$$\text{div} g_\tau = \nabla u_\eta \cdot \eta \text{ on } \partial F.$$ 

Such measures are weak$^*$ dense in $bB(F)^\perp$, i.e., for all $\mu \in bB(F)^\perp$ there exists a sequence $\{g^m\}_{m \in \mathbb{N}} \subset A_0(F)$ such that $\text{div} g^m = \nabla u^m \cdot \eta$ for all $m$ and $\lim_{m \to \infty} \langle f, g^m d\sigma \rangle = \langle f, \mu \rangle$ for all $f \in C(\partial F)^d$.

Moreover, the set of measures $gd\sigma + vds|_\Gamma$, with $g \in A_0(F)$, $\text{div} g_\tau = \nabla u_\eta \cdot \eta$ on $\partial F$ and $vds|_\Gamma \in G(F)$, is a weak$^*$ dense subset of $B(F)^\perp$. Therefore, any $\mu \in B(F)^\perp$ splits into a measure in $bB(F)^\perp$ plus a measure supported in $F$ which annihilates the gradients of functions in $C^1(F)$.

**Proof.** Let $g \in A_0(F)$ and take $\nabla \varphi \in B(F)$. Then,

$$\langle \nabla \varphi, gd\sigma \rangle = \int_{\partial F} (\nabla \varphi)_\tau \cdot g_\tau d\sigma + \int_{\partial F} (\nabla \varphi \cdot \eta) g_\eta d\sigma = -\int_{\partial F} \varphi \text{div} g_\tau d\sigma + \int_{\partial F} (\nabla \varphi \cdot \eta) u_\eta d\sigma.$$ 

The pair $\varphi$ and $u_\eta$ satisfies the statements of Theorem 2.1.9 by Remark 2.1.10, so

$$\langle \nabla \varphi, gd\sigma \rangle = \int_{\partial F} \varphi (\nabla u_\eta \cdot \eta - \text{div} g_\tau) d\sigma.$$
This integral vanishes for all \( \varphi \) such that \( \nabla \varphi \in B(F) \) if and only if \( \text{div} g_\tau = \nabla u_\eta \cdot \eta \) on \( \partial F \).

To prove that such measures \( gd\sigma \) are weak* dense in \( bB(F)^\perp \) it is enough to prove that if \( f \in C(\partial F)^d \) and \( \langle f, gd\sigma \rangle = 0 \) for all such \( g \), then there exists \( \psi \in U_c(F) \) with \( (\nabla \psi)|_{\partial F} = f \).

So, consider \( f \in C(\partial F)^d \) such that

\[
0 = \langle f, gd\sigma \rangle = \int_{\partial F} (f_\tau \cdot g_\tau + f_\eta g_\eta) d\sigma \tag{2.2.2}
\]

for all \( g \in A_0(F) \) such that \( \text{div} g_\tau = \nabla u_\eta \cdot \eta \) on \( \partial F \). By Hodge’s decomposition theorem on the Riemannian manifold \( \partial F \) (see [MC, Lemma 9.1]), \( f_\tau = d\varphi + h_\tau \), where \( \varphi \in C^1(\partial F) \) and \( h_\tau \) is a tangential vector field on \( \partial F \) such that \( \text{div} h_\tau = 0 \). If we take \( g_\eta = 0 \) and \( g_\tau = h_\tau \) in (2.2.2), we obtain

\[
0 = \langle f, gd\sigma \rangle = \int_{\partial F} (d\varphi \cdot g_\tau + h_\tau \cdot g_\tau) d\sigma = \int_{\partial F} (d\varphi \cdot h_\tau + h_\tau \cdot h_\tau) d\sigma
\]

so \( h_\tau = 0 \) and \( f_\tau = d\varphi \). This implies that

\[
\int_{\partial F} f_\tau \cdot g_\tau d\sigma = \int_{\partial F} d\varphi \cdot g_\tau d\sigma = -\int_{\partial F} \varphi \text{div} g_\tau d\sigma = -\int_{\partial F} \varphi (\nabla u_\eta \cdot \eta) d\sigma,
\]

and then (2.2.2) takes the form

\[
\int_{\partial F} f_\eta u_\eta d\sigma = \int_{\partial F} \varphi (\nabla u_\eta \cdot \eta) d\sigma. \tag{2.2.3}
\]

Let \( H \) be a bounded connected component of \( F^c \), and consider a vector field \( g \in A_0(F) \) such that \( g = \eta \) on \( \partial H \) and \( g = 0 \) elsewhere. Then (2.2.3) shows that \( \int_{\partial H} f_\eta d\sigma = 0 \), so we can apply Theorem 2.1.7 (and also Remark 2.1.8 for \( d = 2 \)) to solve the Neumann problem on the complement of \( F \) with boundary data \( f_\eta \). Therefore, there exists a function \( \psi \in C^\infty(F^c) \cap C(\overline{F^c}) \) harmonic in \( F^c \) such that \( \psi|_{\partial F} = f_\eta \). Moreover, we can take \( \psi \) such that \( \nabla \psi(\infty) = 0 \) (\( \psi \) is given by (2.1.2) if \( d = 2 \)).

As we did in Remark 2.1.10, it is easily checked that \( u_\eta \) and \( \psi \) satisfy the statements of Theorem 2.1.9 if \( d \geq 2 \), so we deduce from (2.2.3) that

\[
\int_{\partial F} (\varphi - \psi) (\nabla u_\eta \cdot \eta) d\sigma = 0. \tag{2.2.4}
\]

This equality holds for all \( u_\eta \) harmonic in \( F^c \) and smooth up to \( \partial F \) such that there exists \( g \in A_0(F) \) with \( u_\eta = g \cdot \eta \) and \( \text{div} g_\tau = \nabla u_\eta \cdot \eta \) on \( \partial F \). We are going to see that (2.2.4) implies that \( \varphi - \psi \) is constant on each connected component of \( \partial F \).

Suppose we have a smooth function \( q \) on \( \partial F \) such that \( \int_S q d\sigma = 0 \) for each connected component \( S \) of \( \partial F \). Then, by the maximal de Rham cohomology theorem on the Riemannian
manifold $S$ (see [La1, Theorem 1.1 of Chapter XVIII] for example), the differential form $qd\sigma$ is exact, i.e., there exists a smooth vector field $w$ tangent to $S$ and such that $\text{div} w = q$. On the other hand, the Neumann problem with boundary data $q$ can be solved on $F^c$ by Theorem 2.1.7. Therefore, we can assume that this solution (call it $u$) vanishes at infinity. Summarizing, given the function $q$ we have constructed a smooth vector field $w + u \cdot \eta \in A_0(F)$ such that $\text{div} w = \nabla u \cdot \eta = q$ on $\partial F$.

So, we deduce from (2.2.4) that

$$\int_{\partial F} (\varphi - \psi) q d\sigma = 0 \tag{2.2.5}$$

for all smooth functions $q$ on $\partial F$ such that $\int_S q d\sigma = 0$ for each connected component $S \subset \partial F$.

Therefore, we deduce from (2.2.5) that $\varphi - \psi$ is constant on each component of $\partial F$, and we obtain $(\nabla \psi)_r = d\varphi = f_r$ on $\partial F$. Remember that $\nabla \psi \cdot \eta = f_\eta$, so $\nabla \psi = f$ on $\partial F$. As $f \in C(\partial F)^d$ and $\Delta \psi = 0$ in $F^c$, we have $\psi \in C^1(\overline{F^c})$ because each coordinate of $\nabla \psi$ must be given by the Poisson integral of the corresponding coordinate of $f$. By the Whitney extension theorem (see [Wh], or [Pr, Theorem 8]), we can extend $\psi$ inside $F$ to have $\psi \in C^1(\mathbb{R}^d)$. We have finally obtained $\psi \in U_c(F)$ and $|\nabla \psi|_{\partial F} = f$, so the first part of the theorem is proved.

It only remains to prove that the measures of the type $gd\sigma + vds|_\Gamma$, with $g \in A_0(F)$, $\text{div} g_r = \nabla u_g \cdot \eta$ on $\partial F$ and $vds|_\Gamma \in G(F)$, are a weak* dense subset of $B(F)^\perp$.

That such measures are a subset of $B(F)^\perp$ follows easily from the first part of the theorem and from the fact that $\langle \nabla \phi, vds|_\Gamma \rangle = 0$ for all $\phi \in C^1(F)$ and $vds|_\Gamma \in G(F)$ (the fields coming from a potential have vanishing circulation over closed curves).

To prove that such measures are weak* dense in $B(F)^\perp$ it is enough to prove that if $f \in C(F)^d$ and $\langle f, gd\sigma + vds|_\Gamma \rangle = 0$ for all such $gd\sigma + vds|_\Gamma$, then there exists $\psi \in U_c(F)$ with $|\nabla \psi|_{\partial F} = f$. So, consider $f \in C(F)^d$ such that $\langle f, gd\sigma + vds|_\Gamma \rangle = 0$ for all $gd\sigma + vds|_\Gamma$ with $g \in A_0(F)$, $\text{div} g_r = \nabla u_g \cdot \eta$ on $\partial F$ and $vds|_\Gamma \in G(F)$. By taking $v = 0$, we have seen in the first part of the proof of the theorem that $f = \nabla \psi$ on $\partial F$, for some $\psi \in C^1(\overline{F^c})^d$ with $\Delta \psi = 0$ in $F^c$.

Now, by taking $g = 0$, we see that $\langle f, vds|_\Gamma \rangle = 0$ for all $vds|_\Gamma \in G(F)$. Thus the field $f$ has vanishing circulation over any closed curve, so it is conservative and, hence, it comes from a potential. This potential can be defined continuously up to the boundary because $f \in C(F)^d$. Therefore, there exists a function $\phi \in C^1(F)$ such that $\nabla \phi = f$ in $F$.

Finally, notice that $\nabla \psi = f = \nabla \phi$ on $\partial F$, so the same holds for the tangential components of the gradients. Hence, $\psi - \phi$ is constant in each connected component of $\partial F$. Let $\{H_i\}_{i=1,\ldots,r}$ be the bounded connected components of $F^c$ and let $H_0$ be the unbounded one. Then, $\psi - \phi = C_i$ on $\partial H_i$ for $i = 0, \ldots, r$, where $C_i$ are constants. Define the function

$$\tilde{\psi} := \psi|_{H_0} + (\phi + C_0)|_F + \sum_{i=1}^{r} (\psi + C_0 - C_i)|_{H_i}.$$
2.3. An open question: Some consequences for an affirmative answer

Then, it is easily verified that the function \( \widetilde{\psi} \) belongs to \( C^1(\mathbb{R}^d) \) and it is harmonic outside \( F \), thus it belongs to \( U_c(F) \). Also, \( \nabla \widetilde{\psi} = f \) in \( F \), as desired. The theorem is completely proved.

**Proposition 2.2.3.** For any vector measure \( \mu \in bB(F)^{\perp} \), we have \( \partial_0 F \subseteq \text{supp}(\eta d\sigma_0 + \mu) \).

**Proof.** We will prove the proposition by contradiction. Given a measure \( \mu \in bB(F)^{\perp} \), set \( \nu = \eta d\sigma_0 + \mu \) (so \( \text{supp}\nu \subseteq \partial F \)) and assume that \( \partial_0 F \setminus \text{supp}\nu \neq \emptyset \), so there exists an open ball \( U \) centered at a point of \( \partial_0 F \) such that \( |\nu|(U) = 0 \). Consider the function

\[
w(x) = \int \nabla \phi_d(x - y) \, d\nu(y)
\]

for \( x \in \mathbb{R}^d \setminus \text{supp}\nu \). Since all the components of \( \nabla \phi_d \) are harmonic outside the origin, we have \( \Delta w = 0 \) outside the support of \( \nu \), and in particular, in \( U \). Notice that, since \( \nu \) is a finite vector measure, \( w(\infty) = 0 \). On the other hand, given \( x \in \text{int} F \), we can easily modify the function \( \phi_d(x - \cdot) \) in a small neighborhood \( V \) of \( x \) with \( \overline{V} \subseteq \text{int} F \) to obtain a function \( \psi_x \in U_c(F) \) such that \( \psi_x = \phi_d(x - \cdot) \) outside \( V \). Then, since \( \mu \in bB(F)^{\perp} \),

\[
w(x) = \int_{\partial_0 F} \nabla \phi_d(x - y) \eta(y) \, d\sigma_0(y) + \int_{\partial F} \nabla \phi_d(x - y) \, d\mu(y) = 1 - \int_{\partial F} \nabla \psi_x \, d\mu = 1.
\]

Therefore, \( w \) is constantly equal to 1 in \( \text{int} F \), harmonic outside \( \text{supp}\nu \), and vanishes at infinity. But the (open) unbounded connected component of \( (\text{supp}\nu)^c \) intersects \( \text{int} F \) through \( U \), so \( w \) must be equal to one in the whole component, which contradicts the fact that \( w(\infty) = 0 \). The proposition is proved.

2.3 An open question: Some consequences for an affirmative answer

We are very interested in the following question, because a positive answer yields new properties on the \( \kappa \) and \( \kappa_c \) capacities. Unfortunately, as we said at the beginning of the chapter, we have not been able to give an answer to it.

**Question 2.3.1.** Is it true that, for any \( F \subseteq \mathbb{R}^d \) which is the closure of a bounded open set with smooth boundary, there exists a Garabedian measure of \( F \) which is supported in \( \partial F \)?

**Theorem 2.3.2.** Let \( F \subseteq \mathbb{R}^d \) be the closure of a bounded open set with smooth boundary. If Question 2.3.1 has an affirmative answer for this concrete \( F \), then

\[
\kappa_c(F) = \min \left\{ \| \eta d\sigma_0 + \mu \|_{\partial_0 F} : \mu \in B(F)^{\perp}, \text{supp}\mu \subseteq \partial_0 F \right\}.
\]

Moreover, there exists a Garabedian measure \( \nu \) of \( F \) such that \( \text{supp}\nu = \partial_0 F \).
Denote by $V$ the unbounded connected component of $E^c$ and consider the bounded open set $V' = V^c$. Take an increasing sequence of open sets $\{V'_m\}_{m \in \mathbb{N}}$ such that

$$V'_m \subset V', \quad \overline{V'_m} \subset V'_{m+1}, \quad \partial V'_m \subset F^1_m, \quad \bigcup_{m \in \mathbb{N}} V'_m = V'$$

and with smooth boundary, for all $m$. Finally, define the sequence $\{F^2_m := F^1_m \setminus V'_m\}_{m \in \mathbb{N}}$. By construction, the sequence $\{F^2_m\}_{m \in \mathbb{N}}$ is a decreasing sequence of closures of bounded open sets with smooth boundary. Observe also that $\partial_o F^1_m = \partial_o F^2_m$ for all $m \in \mathbb{N}$.

Now, if $x \in \partial_o E = \partial V$, then $x \in F^2_m$ for all $m \in \mathbb{N}$. On the other hand, if $x \in F^2_m \subset F^1_m$ for all $m \in \mathbb{N}$, then $x \in \overline{V} \cap E$, so $x \in \partial_o E$. Therefore, we have proved that $\bigcap_{m \in \mathbb{N}} F^2_m = \partial_o E$. 

**Proof.** The second statement of the theorem is a direct consequence of the first one and Proposition 2.2.3. By Theorem 2.2.1, if Question 2.3.1 has an affirmative answer, we have $\kappa_c(F) = \min \{ ||\eta d\sigma_o + \mu|_{\partial F} : \mu \in bB(F)^\perp \}$. Thus, it only remains to check that the minimum is attained on a measure $\mu \in bB(F)^\perp$ with $\text{supp}\mu \subset \partial_o F$ (recall that $\text{supp}\sigma_o \subset \partial_o F$).

By Theorem 2.2.2, for every $\mu \in bB(F)^\perp$ we can find a sequence $h_m \in A_0(F)$ such that $h_m d\sigma \in bB(F)^\perp$ weak* tends to $\mu$. Since the connected components of $\partial F$ are a finite number of disjoint compact sets, we see that $h_m \chi_S d\sigma$ weak* tends to $\mu|_S$ for every connected component $S$ of $\partial F$. In particular, $h_m \chi_{\partial_o F} d\sigma$ weak* tends to $\mu|_{\partial_o F}$.

Let $h^\eta_m$ be the normal component of $h_m$, and let $u_m$ be the harmonic extension of $h^\eta_m$ in $F^c \cup \{\infty\}$. Since $h_m d\sigma \in bB(F)^\perp$, $\text{div} h^\tau_m = \nabla u_m \cdot \eta$ on $\partial F$ by Theorem 2.2.2. Let $v_m$ be the harmonic extension of $h^\eta_m \chi_{A_F}$ in $F^c \cup \{\infty\}$, thus $v_m = 0$ in every bounded connected component of $F^c$ and $v_m = u_m$ in the unbounded one. Evidently, $\text{div}(h^\tau_m \chi_{A_F}) = \nabla v_m \cdot \eta$ on $\partial F$ and $h_m \chi_{\partial_o F} \in A_0(F)$, so $h_m \chi_{\partial_o F} d\sigma \in bB(F)^\perp$ by Theorem 2.2.2 again. Since $h_m \chi_{\partial_o F} d\sigma$ weak* tends to $\mu|_{\partial_o F}$ and $h_m \chi_{\partial_o F} d\sigma \in bB(F)^\perp$ for all $m$, we conclude that $\mu|_{\partial_o F} \in bB(F)^\perp$. We also clearly have $\|\eta d\sigma_o + \mu|_{\partial_o F}\|_{\partial F} \leq \|\eta d\sigma_o + \mu\|_{\partial F}$. To summarize, for every measure $\mu \in bB(F)^\perp$ we have seen that $\mu|_{\partial_o F} \in bB(F)^\perp$ and that

$$\|\eta d\sigma_o + \mu|_{\partial_o F}\|_{\partial F} = \|\eta d\sigma_o + \mu|_{\partial_o F}\|_{\partial F} \leq \|\eta d\sigma_o + \mu\|_{\partial F},$$

so the theorem is proved. 

**Theorem 2.3.3.** If Question 2.3.1 has an affirmative answer, then $\kappa(E) = \kappa(\partial_o E)$ for every compact set $E \subset \mathbb{R}^d$.

**Proof.** Since $\kappa$ is non decreasing set function, $\kappa(E) \geq \kappa(\partial_o E)$. In order to prove the converse inequality, let $\{F^1_m\}_{m \in \mathbb{N}}$ be a sequence of closures of smooth neighborhoods of $E$ collapsing to $E$, i.e.

$$E \subset F^1_{m+1} \subset \text{int} F^1_m : \bigcap_{m \in \mathbb{N}} F^1_m = E.$$ 

Denote by $V$ the unbounded connected component of $E^c$ and consider the bounded open set $V' = V^c$. Take an increasing sequence of open sets $\{V'_m\}_{m \in \mathbb{N}}$ such that

$$V'_m \subset V', \quad V'_m \subset V'_{m+1}, \quad \partial V'_m \subset F^1_m, \quad \bigcup_{m \in \mathbb{N}} V'_m = V'$$

and with smooth boundary, for all $m$. Finally, define the sequence $\{F^2_m := F^1_m \setminus V'_m\}_{m \in \mathbb{N}}$. By construction, the sequence $\{F^2_m\}_{m \in \mathbb{N}}$ is a decreasing sequence of closures of bounded open sets with smooth boundary. Observe also that $\partial_o F^1_m = \partial_o F^2_m$ for all $m \in \mathbb{N}$.

Now, if $x \in \partial_o E = \partial V$, then $x \in F^2_m$ for all $m \in \mathbb{N}$. On the other hand, if $x \in F^2_m \subset F^1_m$ for all $m \in \mathbb{N}$, then $x \in \overline{V} \cap E$, so $x \in \partial_o E$. Therefore, we have proved that $\bigcap_{m \in \mathbb{N}} F^2_m = \partial_o E$. 


By [Pr, Lemma 2.2(7)],
\[ \lim_{m \to \infty} \kappa(F^1_m) = \kappa(E), \quad \text{and} \quad \lim_{m \to \infty} \kappa(F^2_m) = \kappa(\partial_o E). \]

Let \( \eta_m \) be the outward unit normal vector on \( \partial_o F^1_m \) and \( d\sigma_m \) the surface measure on \( \partial_o F^1_m \). We claim that a measure \( \mu \) supported in \( \partial_o F^1_m \) belongs to \( B(F^1_m)^\perp \) if and only if it belongs to \( B(F^2_m)^\perp \). On one hand, any \( \mu \in B(F^1_m)^\perp \) supported in \( \partial_o F^1_m \) belongs to \( B(F^2_m)^\perp \) because \( F^2_m \subset F^1_m \) and \( \partial_o F^1_m = \partial_o F^2_m \). On the other hand, suppose that \( \mu \in B(F^2_m)^\perp \). Given \( \nabla \varphi \in B(F^1_m) \) with \( \varphi \in U_c(F^1_m) \), we can modify \( \varphi \) to obtain a function \( \psi \in U_c(F^2_m) \) such that \( \psi = 0 \) in the bounded connected components of \( (F^2_m)^c \) and \( \psi = \varphi \) in the unbounded one, thus \( \nabla \varphi = \nabla \psi \) on \( \partial_o F^1_m \). Then, \( \nabla \psi \in B(F^2_m) \) and \( \langle \nabla \varphi, \mu \rangle = \langle \nabla \psi, \mu \rangle = 0 \) because \( \mu \in B(F^2_m)^\perp \) is supported in \( \partial_o F^1_m \). We are done with the “if and only if” claim.

Applying Theorem 2.3.2 to \( F^1_m \) and \( F^2_m \), and by the observation above,
\[ \kappa_c(F^1_m) = \min \{ \| \eta_m d\sigma_m + \mu \|_{\partial_o F^1_m} : \mu \in B(F^1_m)^\perp, \text{supp} \mu \subset \partial_o F^1_m \} = \min \{ \| \eta_m d\sigma_m + \mu \|_{\partial_o F^2_m} : \mu \in B(F^2_m)^\perp, \text{supp} \mu \subset \partial_o F^2_m \} = \kappa_c(F^2_m). \]

By definition, \( \kappa_c \) is also monotone and \( \kappa_c \leq \kappa \) and, by [Pr, Lemma 2.2(1)], both capacities coincide on open sets. So, we have
\[ \kappa(F^1_{m+1}) = \kappa(\text{int} F^1_m) = \kappa_c(\text{int} F^1_m) \leq \kappa_c(F^1_m) = \kappa_c(F^2_m) = \kappa(F^2_m). \]

The theorem is proved by letting \( m \) tend to infinity. \( \square \)

**Theorem 2.3.4.** Let \( f \) be a real continuous function defined on the cube \( Q_0 = [0, \ell]^{d-1} \subset \mathbb{R}^{d-1} \) and let \( \Gamma = \{(x, f(x)) \in \mathbb{R}^d : x \in Q_0\} \) be the graph of \( f \). Then, if Question 2.3.1 has an affirmative answer, there exists a constant \( C > 0 \) depending only on \( d \) such that \( C\ell^{d-1} \leq \kappa(\Gamma) \).

**Proof.** The proof is based on Theorem 2.3.3 and the semiadditivity of \( \kappa \). Starting from \( Q_0 \), consider a decomposition of \( \mathbb{R}^{d-1} \) into cubes \( Q_i \) of side length \( \ell \) and with disjoint interiors. By doing reflections with respect to the sides of the cubes \( Q_i \), we can extend \( f \) to a function \( \tilde{f} \) continuous on \( \mathbb{R}^{d-1} \) and such that \( \tilde{f} \) in \( Q_i \) is a reflection of \( f \) in \( Q_0 \).

Let \( Q^m \) be a cube in \( \mathbb{R}^{d-1} \) of side length \( m\ell \) made by the union of \( m^{d-1} \) cubes \( Q_i \). Let \( \Gamma_m \) be the graph of the function \( \tilde{f} \) on \( Q^m \), i.e., \( \Gamma_m = \{(x, \tilde{f}(x)) \in \mathbb{R}^d : x \in Q^m\} \), and consider its translation \( \Gamma'_m = \{(x, \tilde{f}(x) + 4\|f\|_{Q_0}) \in \mathbb{R}^d : x \in Q^m\} \). Clearly, the sets \( \Gamma_m \) and \( \Gamma'_m \) do not intersect. Moreover, they are separated by the set \( P_m = \{(x, 2\|f\|_{Q_0}) : x \in Q^m\} \).

Finally, let \( E_m \) be the region enclosed by \( \Gamma_m \), \( \Gamma'_m \) and the \( 2(d-1) \) pieces of vertical hyperplanes of \( \mathbb{R}^d \) that join the edges of \( \Gamma_m \) and \( \Gamma'_m \) (when \( d = 2 \), this vertical hyperplanes are just two segments joining the endpoints of \( \Gamma_m \) and \( \Gamma'_m \)). Roughly speaking, \( E_m \) is a kind
of \(d\)-dimensional rectangle which has \(\Gamma_m\) as the bottom side and \(\Gamma'_m\) as the top side. By construction, \(E_m\) is a compact set that contains \(P_m\), thus \(\kappa(E_m) \geq \kappa(P_m)\).

In [Pr, Lemma 2.2(8)] it is shown that for any \(L^{d-1}\)-measurable set \(E\) lying in some hyperplane of \(\mathbb{R}^d\), \(\kappa_c(E) = 0\) and \(A_1 L^{d-1}(E) \leq \kappa(E) \leq A_2 L^{d-1}(E)\), where \(A_1\) and \(A_2\) are two positive constants that depend only on \(d\) (this means that \(\kappa\) behaves like \(L^{d-1}\) on subsets of hyperplanes). In particular, we have

\[
\kappa(E_m) \geq \kappa(P_m) = C(m\ell)^{d-1}. 
\]  

Applying Theorem 2.3.3, the countable semiadditivity of \(\kappa\), and [Pr, Lemma 2.2(8)], we have

\[
\kappa(E_m) = \kappa(\partial_o E_m) \leq C\big(\kappa(\Gamma_m) + \kappa(\Gamma'_m) + 20(d - 1)\|f\|_{Q_0(m\ell)}^{d-2}\big). 
\]  

By the construction of \(\Gamma_m\) and the semiadditivity of \(\kappa\), \(\kappa(\Gamma'_m) = \kappa(\Gamma_m) \leq Cm^{d-1}\kappa(\Gamma)\), so

\[
(2.3.2) \quad \kappa(E_m) \leq C\big(2m^{d-1}\kappa(\Gamma) + 20(d - 1)\|f\|_{Q_0(m\ell)}^{d-2}\big). 
\]

Combining (2.3.1) and (2.3.3), we get

\[
\kappa(\Gamma) \geq \frac{Cm^{d-1}\ell^{d-1} - C'\|f\|_{Q_0(m\ell)}^{d-2}}{2m^{d-1}} = C\ell^{d-1} - C'\frac{\|f\|_{Q_0}^{d-2}}{m},
\]

where \(C' > 0\) is an absolute constant which only depends on the dimension \(d\). Letting \(m \to \infty\), we obtain \(\kappa(\Gamma) \geq C\ell^{d-1}\), and the theorem is proved.

\[\Box\]

**Remark 2.3.5.** Under an affirmative answer to Question 2.3.1, one can show that \(\kappa(E) \geq C\text{diam}(E)\) for any continuum \(E \subset \mathbb{R}^2\) by using the same ideas as in the proof of Theorem 2.3.4. One starts by choosing two points \(a, b \in E\) such that \(\text{diam}(E) = |b - a|\) and assuming that these points belong to the real axis in \(\mathbb{R}^2\). Then, one extends the set \(E\) by symmetries along the real axis as we did before. The rest of the proof remains the same.

The inequality \(\kappa(E) \geq C\text{diam}(E)\) for a continuum \(E \subset \mathbb{R}^2\) was stated as an open question in [Pr, Problem 2.6] and was first proved by P. Jones by using the notion of curvature of a measure and other capacities called \(\gamma_+\) and \(\kappa_+\) (see [Pa], [To6], and [Vo]).
Chapter 3

Variation and oscillation for singular integrals with odd kernel on Lipschitz graphs: Smooth truncation

The $\rho$-variation and oscillation for martingales and some families of operators have been studied in many recent papers on probability, ergodic theory, and harmonic analysis (see [Lé], [Bo], [JKRW], [CJRW1], and [JSW], for example). The purpose of this chapter is to establish some new results concerning the $\rho$-variation and oscillation for families of singular integral operators defined on Lipschitz graphs. In particular, our results include the $L^p$ boundedness of the $\rho$-variation and the oscillation for the Cauchy transform and the $n$-dimensional Riesz transform on Lipschitz graphs (with slope smaller than 1 if we consider truncations given by characteristic functions of balls), for $1 < p < \infty$ and $\rho > 2$.

Given a Borel measure $\mu$ in $\mathbb{R}^d$, one defines the $n$-dimensional Riesz transform of a function $f \in L^1(\mu)$ by $R^\mu f(x) = \lim_{\epsilon \searrow 0} R^\mu_\epsilon f(x)$ (whenever the limit exists), where

$$R^\mu_\epsilon f(x) = \int_{|x-y|>\epsilon} \frac{x-y}{|x-y|^{n+1}} f(y) \, d\mu(y), \quad x \in \mathbb{R}^d.$$  

When $d = 2$ (i.e., $\mu$ is a Borel measure in $\mathbb{C}$), one defines the Cauchy transform of $f \in L^1(\mu)$ by $C^\mu f(x) = \lim_{\epsilon \searrow 0} C^\mu_\epsilon f(x)$ (whenever the limit exists), where

$$C^\mu_\epsilon f(x) = \int_{|x-y|>\epsilon} \frac{f(y)}{x-y} \, d\mu(y), \quad x \in \mathbb{C}.$$  

To avoid the problem of existence of the preceding limits, it is useful to consider the maximal operators $R^\mu_\epsilon f(x) = \sup_{\epsilon > 0} |R^\mu_\epsilon f(x)|$ and $C^\mu_\epsilon f(x) = \sup_{\epsilon > 0} |C^\mu_\epsilon f(x)|$.

The Cauchy and Riesz transforms are two very important examples of singular integral operators with a Calderón-Zygmund kernel. The kernels $K : \mathbb{R}^d \setminus \{0\} \rightarrow \mathbb{R}$ that we consider
for definiteness and they are not important, we have chosen these specific ones because they satisfy
\[ |K(x)| \leq \frac{C}{|x|^n}, \quad |\partial_{x_i} K(x)| \leq \frac{C}{|x|^{n+1}} \quad \text{and} \quad |\partial_{x_i} \partial_{x_j} K(x)| \leq \frac{C}{|x|^{n+2}}, \quad (3.0.1) \]
for all \(1 \leq i, j \leq d\) and \(x = (x^1, \ldots, x^d) \in \mathbb{R}^d \setminus \{0\}\), where \(0 < n < d\) is some integer and \(C > 0\) is some constant; and moreover \(K(-x) = -K(x)\) for all \(x \neq 0\) (i.e. \(K\) is odd). Notice that the \(n\)-dimensional Riesz transform corresponds to the vector kernel \((x^1, \ldots, x^d)/|x|^{n+1}\), and the Cauchy transform to \((x^1, -x^2)/|x|^2\) (so, we may consider \(K\) to be any scalar component of these vector kernels).

Given an odd kernel \(K\) satisfying (3.0.1) and a finite Borel measure \(\mu\) in \(\mathbb{R}^d\), for each \(\epsilon > 0\), we consider the \(\epsilon\)-truncated operator
\[ T_{\epsilon} \mu(x) = \int_{|x-y|>\epsilon} K(x-y) \, d\mu(y), \quad x \in \mathbb{R}^d, \]
and then we set \(T_{\epsilon} \mu(x) = \sup_{\epsilon > 0} |T_{\epsilon} \mu(x)|\) and \(T \mu(x) = \lim_{\epsilon \to 0} T_{\epsilon} \mu(x)\), whenever the limit makes sense. Finally, given \(f \in L^1(\mu)\), we define \(T_{\epsilon}^n f(x) := T_{\epsilon} (f \mu)(x)\), \(T^n f(x) := T(f \mu)(x)\) and \(T_{\epsilon}^n f(x) := T_{\epsilon} (f \mu)(x)\). Thus, for a suitable choice of \(K\), the operator \(T^n\) coincides with the Cauchy or Riesz transforms.

Besides the operator \(T_{\epsilon}\) defined above, one can consider other \(\epsilon\)-truncated variants that we proceed to define. First we need some additional notation. Given \(x = (x^1, \ldots, x^d) \in \mathbb{R}^d\), we use the notation \(\tilde{x} := (x^1, \ldots, x^n) \in \mathbb{R}^n\). Let \(\varphi_\mathbb{R} : [0, \infty) \to [0, \infty)\) be a non decreasing \(C^2\) function such that \(\chi_{[3\sqrt{n}, \infty)} \leq \varphi_\mathbb{R} \leq \chi_{[2.1\sqrt{n}, \infty)}\) (the numbers \(3\sqrt{n}\) and \(2.1\sqrt{n}\) are chosen just for definiteness and they are not important, we have chosen these specific ones because they simplify a bit the future computations). Given \(\epsilon > 0\) and \(x \in \mathbb{R}^d\), we denote
\[ \varphi_\epsilon(x) := \varphi_\mathbb{R}(|x|/\epsilon) \quad \text{and} \quad \tilde{\varphi}_\epsilon(x) := \varphi_\mathbb{R}(|\tilde{x}|/\epsilon), \]
\[ \chi_\epsilon(x) := \chi_{(1, \infty)}(|x|/\epsilon) \quad \text{and} \quad \tilde{\chi}_\epsilon(x) := \chi_{(1, \infty)}(|\tilde{x}|/\epsilon). \]

**Definition 3.0.6 (family of truncations).** We consider the following families of functions
\[ \varphi := \{\varphi_\epsilon\}_{\epsilon > 0}, \quad \tilde{\varphi} := \{\tilde{\varphi}_\epsilon\}_{\epsilon > 0}, \quad \chi := \{\chi_\epsilon\}_{\epsilon > 0}, \quad \tilde{\chi} := \{\tilde{\chi}_\epsilon\}_{\epsilon > 0}. \]
We say that a family of functions \(\omega := \{\omega_\epsilon\}_{\epsilon > 0}\) is a family of truncations if \(\omega \in \{\varphi, \tilde{\varphi}, \chi, \tilde{\chi}\}\).

Let \(\omega := \{\omega_\epsilon\}_{\epsilon > 0}\) be a family of truncations. Given a kernel \(K\) as above, \(x \in \mathbb{R}^d, \epsilon > 0,\) and a finite Borel measure \(\mu\), we consider
\[ (K \omega_\epsilon \ast \mu)(x) := \int \omega_\epsilon(x-y) K(x-y) \, d\mu(y). \]
We also denote \((K \omega \ast \mu)(x) := \{(K \omega_\epsilon \ast \mu)(x)\}_{\epsilon > 0}\). Finally, given \(f \in L^1(\mu)\), we define
\[ T_{\omega_\epsilon}^n f(x) := (K \omega_\epsilon \ast (f \mu))(x), \quad T_{\omega_\epsilon}^n f(x) := \sup_{\epsilon > 0} |T_{\omega_\epsilon}^n f(x)|, \quad T_{\omega_\epsilon}^n f(x) := \lim_{\epsilon \to 0} T_{\omega_\epsilon}^n f(x)\] (whenever
the limit makes sense), and \( \mathcal{T}_ω^μ f(x) := \{T_ω^μ f(x)\}_{ε>0} \). For the particular case of \( ω = χ \), notice that \( T_χ^μ f = T_χ^μ f \), thus we obtain the truncated Cauchy and Riesz transforms taking a suitable kernel \( K \).

**Definition 3.0.7** (\( ρ \)-variation and oscillation). Let \( I \) be a subset of \( \mathbb{R} \) (in this chapter, we will always have \( I = (0, \infty) \) or \( I = \mathbb{Z} \)), and let \( F := \{F_ε\}_{ε∈I} \) be a family of functions defined on \( \mathbb{R}^d \). Given \( ρ > 0 \), the \( ρ \)-variation of \( F \) at \( x ∈ \mathbb{R}^d \) is defined by

\[
V_ρ(F)(x) := \sup_{\{ε_m\}} \left( \sum_{m∈\mathbb{Z}} |F_{ε_{m+1}}(x) - F_{ε_m}(x)|^ρ \right)^{1/ρ},
\]

where the pointwise supremum is taken over all decreasing sequences \( \{ε_m\}_{m∈\mathbb{Z}} ⊂ I \). Fix a decreasing sequence \( \{r_m\}_{m∈\mathbb{Z}} ⊂ I \). The oscillation of \( F \) at \( x ∈ \mathbb{R}^d \) is defined by

\[
O(F)(x) := \sup_{\{ε_m\}, \{δ_m\}} \left( \sum_{m∈\mathbb{Z}} |F_{ε_m}(x) - F_{δ_m}(x)|^2 \right)^{1/2},
\]

where the pointwise supremum is taken over all sequences \( \{ε_m\}_{m∈\mathbb{Z}} ⊂ I \) and \( \{δ_m\}_{m∈\mathbb{Z}} ⊂ I \) such that \( r_{m+1} ≤ ε_m ≤ δ_m ≤ r_m \) for all \( m ∈ \mathbb{Z} \).

In this chapter we are interested in studying the \( ρ \)-variation and oscillation of the families \( \mathcal{T}_ω^μ f \), for the truncations \( ω \) introduced above. That is, we will deal with

\[
(V_ρ \circ \mathcal{T}_ω^μ) f(x) := V_ρ(\mathcal{T}_ω^μ f)(x) = V_ρ(Kω * (fμ))(x) \quad \text{and}
\]

\[
(O \circ \mathcal{T}_ω^μ) f(x) := O(\mathcal{T}_ω^μ f)(x) = O(Kω * (fμ))(x),
\]

for a Borel measure \( μ \) and \( f ∈ L^1(μ) \). Although it is not clear from the definitions, these operators are \( μ \)-measurable (see [CJRW1], [JSW]).

Given \( E ⊂ \mathbb{R}^d \), we denote by \( \mathcal{H}_E^d \) the \( n \)-dimensional Hausdorff measure restricted to \( E \). Let \( Γ := \{x ∈ \mathbb{R}^d : x = (\bar{x}, A(\bar{x}))\} \) be the graph of a Lipschitz function \( A : \mathbb{R}^n → \mathbb{R}^{d-n} \) with Lipschitz constant \( \text{Lip}(A) \). Let \( H^1(\mathcal{H}_Γ^d) \) and \( \text{BMO}(\mathcal{H}_Γ^d) \) be the (atomic) Hardy space and the space of functions with bounded mean oscillation, respectively, with respect to the measure \( \mathcal{H}_Γ^d \). The following is our main result.

**Main Theorem 3.0.1.** Let \( ρ > 2 \), let \( K \) be a kernel satisfying (3.0.1), let \( ω \) be a family of truncations, and set \( μ := \mathcal{H}_Γ^d \). If \( ω \in \{φ, \tilde{φ}, χ\} \), the operators \( V_ρ \circ \mathcal{T}_ω^μ \) and \( O \circ \mathcal{T}_ω^μ \) are bounded

- in \( L^p(μ) \) for \( 1 < p < ∞ \),
- from \( L^1(μ) \) to \( L^{1,∞}(μ) \), and
- from \( L^{∞}(μ) \) to \( \text{BMO}(μ) \).

The same holds if \( ω = χ \) and \( \text{Lip}(A) < 1 \). Furthermore, if \( ω \in \{φ, \tilde{φ}\} \), the operators \( V_ρ \circ \mathcal{T}_ω^μ \) and \( O \circ \mathcal{T}_ω^μ \) are also bounded from \( H^1(μ) \) to \( L^1(μ) \). In all the cases above, the norm of \( O \circ \mathcal{T}_ω^μ \) is bounded independently of the sequence that defines \( O \).
As remarked above, the theorem applies to the particular cases of the Cauchy transform (with $d = 2, n = 1$) and the $n$-dimensional Riesz transforms on $n$-dimensional Lipschitz graphs in $\mathbb{R}^d$.

As we will see in Chapter 5, the $L^2(\mu)$ boundedness of $V_\rho \circ T_\chi^\mu$ and $O \circ T_\chi^\mu$ also holds without the assumption $\operatorname{Lip}(A) < 1$. However, we have not been able to prove this with the techniques used in this chapter for the rest of families of truncations (see Remark 4.1.3).

Let us recall that the $L^2(H^1_\Gamma)$ boundedness of the Cauchy transform on Lipschitz graphs $\Gamma \subset \mathbb{C}$ with slope small enough was proved by A. P. Calderón in his celebrated paper [Ca]. The $L^2$ boundedness on Lipschitz graphs in full generality was proved later on by R. Coifman, A. McIntosh, and Y. Meyer [CMM].

Consider the Cauchy kernel $K(z) = 1/z$ ($z \in \mathbb{C}$), and set $\mu := H^1_\Gamma$, so $C_\mu = T_\chi^\mu$. By standard Calderón-Zygmund theory (namely, Cotlar’s inequality), the $L^2(\mu)$ boundedness of the Cauchy transform $C^\mu$ is equivalent to the $L^2(\mu)$ boundedness of the maximal operator $C_\mu^\ast$. Let $M^\mu$ denote the Hardy-Littlewood maximal operator with respect to the measure $\mu$. It is easy to check that, for $f \in L^1(\mu)$ with compact support, there exists some constant $C_0 > 0$ such that

$$C_\mu^\ast f(x) \leq T_\chi^\mu f(x) + C_0 M^\mu f(x) \leq (V_\rho \circ T_\chi^\mu) f(x) + C_0 M^\mu f(x)$$

for all $\epsilon > 0$, thus $(V_\rho \circ T_\chi^\mu) + C_0 M^\mu$ controls the maximal operator $C_\mu^\ast$ and, in this sense, Theorem 3.0.1 (together with the known $L^p(\mu)$ boundedness of $M^\mu$) strengthens the results of [Ca] and [CMM]. Analogous conclusions hold for the $n$-dimensional Riesz transform and the maximal operator $R^\mu_\mu$.

Concerning the background on the $\rho$-variation and oscillation, a fundamental result is Lépingle’s inequality [Lé], from which the $L^p$ boundedness of the $\rho$-variation and oscillation for martingales follows, for $\rho > 2$ and $1 < p < \infty$ (see Theorem 3.1.4 below for more details). From this result on martingales, one deduces that the $\rho$-variation and oscillation are also bounded in $L^p$ for the averaging operators (also called differentiation operators, see [JKRW]):

$$D_\epsilon f(x) = \frac{1}{|B(x, \epsilon)|} \int_{B(x, \epsilon)} f(y) \, dy, \quad x \in \mathbb{R}. \quad (3.0.2)$$

As far as we know, the first work dealing with the $\rho$-variation and oscillation for singular integral operators is the one of J. Campbell, R. L. Jones, K. Reinhold and M. Wierdl [CJRW1], where the $L^p$ and weak $L^1$ boundedness of the $\rho$-variation (for $\rho > 2$) and oscillation for the Hilbert transform was proved. Recall that, for $f \in L^p(\mathbb{R})$ and $x \in \mathbb{R},$

$$H_\epsilon f(x) = \frac{1}{\pi} \int_{|x-y|>\epsilon} \frac{1}{x-y} f(y) \, dy.$$
and then the Hilbert transform of \( f \) is defined by \( Hf(x) = \lim_{\varepsilon \to 0} H_\varepsilon f(x) \), whenever the limit exists. Later on, there appeared other papers showing the \( L^p \) boundedness of the \( \rho \)-variation and oscillation for singular integrals in \( \mathbb{R}^d \) ([CJRW2]), with weights ([GT]), or for other operators such as the spherical averaging operator or singular integral operators on parabolas ([JSW]). Finally, we remark that, very recently, the case of the Carleson operator has been considered too ([LT], [OSTTW]).

Notice that the Hilbert transform is one of the simplest examples where Theorem 3.0.1 applies (one sets \( \Gamma = \mathbb{R} \), i.e., \( A \equiv 0 \)), and so one obtains a new proof of the \( L^p \) boundedness of the \( \rho \)-variation and oscillation for the Hilbert transform. In the original proof in [CJRW1], a key ingredient was the following classical identity, which follows via the Fourier transform:

\[
Q_\varepsilon = P_\varepsilon * H,
\]

where \( P_\varepsilon \) is the Poisson kernel and \( Q_\varepsilon \) is the conjugated Poisson kernel. Using this identity and the close relationship between the operators \( Q_\varepsilon \) and \( H_\varepsilon \), Campbell et al. derived the \( L^p \) boundedness of the \( \rho \)-variation and oscillation for the Hilbert transform from the one of the family \( \{D_\varepsilon (H f)\}_{\varepsilon > 0} \), where \( D_\varepsilon \) is the averaging operator in (3.0.2) (notice that \( P_\varepsilon \) can be written as a convex combination of operators \( D_\delta \), \( \delta > 0 \)).

In most of the previous results concerning \( \rho \)-variation and oscillation of families of operators from harmonic analysis, the Fourier transform is a fundamental tool. However, this is not useful in order to prove Theorem 3.0.1, since the graph \( \Gamma \) is not invariant under translations in general. Moreover, even for the Cauchy transform, there is no formula like (3.0.3), which relates the truncations of a singular integral operator with an averaging operator applied to a singular integral operator, when \( \Gamma \) is a general Lipschitz graph.

The main ingredients of our proof of Theorem 3.0.1 are the known results on the \( \rho \)-variation and oscillation for martingales (Lépingle’s inequality [Lé]) and a multiscale analysis which stems from the geometric proof of the \( L^2 \) boundedness of the Cauchy transform on Lipschitz graphs by P. W. Jones [Jn1] and his celebrated work [Jn2] on quantitative rectifiability in the plane, using the so-called \( \beta \) coefficients. Some of the techniques in these papers were further developed in higher dimensions by David and Semmes (see [DS1] and [DS2]) for Ahlfors-David regular sets. More recently, in [To11] some coefficients denoted by \( \alpha \), in the spirit of the Jones’ \( \beta \)’s, were introduced, and they were shown to be useful for the study of the \( L^p \)-boundedness of Calderón-Zygmund operators on Lipschitz graphs and on uniformly rectifiable sets (see the definition below Theorem 3.0.8). In this chapter, the \( \alpha \) and \( \beta \) coefficients play a fundamental role.

Let us remark that Lépingle’s inequality, which asserts the \( L^p \) boundedness of the \( \rho \)-variation of martingales, fails if one assumes \( \rho \leq 2 \) (see [Qi] and [JW], for example). Moreover, this fact can be brought to the \( \rho \)-variation of averaging operators and singular integral
operators, thus it is essential to assume \( \rho > 2 \) in Theorem 3.0.1. Analogous conclusions hold if one replaces the \( \ell^2 \)-norm by and \( \ell^p \)-norm with \( \rho < 2 \) in the definition of \( \mathcal{O} \). See [CJRW1], or [AJS] for the case of martingales.

Concerning the applications of Theorem 3.0.1, it is easily seen that the \( L^p \) boundedness of \( \mathcal{V}_\rho \circ \mathcal{T}_\omega^\mu \) yields a new proof of the existence of the principal values \( T_\mu^\omega f(x) = \lim_{r \to 0} T_\mu^\omega f(x) \) for all \( f \in L^p(\mu) \) and almost all \( x \in \Gamma \), without using a dense class of functions in \( L^p(\mu) \) (as in the classical proof). Moreover, from Theorem 3.0.1 one also gets some information on the speed of convergence. In fact, a classical result derived from variational inequalities is the boundedness of the \( \lambda \)-jump operator \( N_\lambda \circ \mathcal{T}_\omega^\mu \) and the \( (a, b) \)-upcrossings operator \( N^b_a \circ \mathcal{T}_\omega^\mu \). Given \( \lambda > 0 \), \( f \in L^1_{\text{loc}}(\mu) \) and \( x \in \mathbb{R}^d \), one defines \( (N_\lambda \circ \mathcal{T}_\omega^\mu)(f)(x) \) as the supremum of all integers \( N \) for which there exists \( 0 < \epsilon_1 < \delta_1 \leq \epsilon_2 < \delta_2 \leq \cdots \leq \epsilon_N < \delta_N \) so that

\[
|T^\mu_{\omega, i} f(x) - T^\mu_{\omega, j} f(x)| > \lambda
\]

for each \( i = 1, \ldots, N \). Similarly, given \( a < b \), one defines \( (N^b_a \circ \mathcal{T}_\omega^\mu)(f)(x) \) to be the supremum of all integers \( N \) for which there exists \( 0 < \epsilon_1 < \delta_1 \leq \epsilon_2 < \delta_2 \leq \cdots \leq \epsilon_N < \delta_N \) so that \( T^\mu_{\omega, i} f(x) < a \) and \( T^\mu_{\omega, j} f(x) > b \) for each \( i = 1, \ldots, N \). Using Theorem 3.0.1 one obtains (by the same arguments as in [CJRW1, Theorem 1.3 and Corollary 7.1]) the following:

**Theorem 3.0.8.** Let \( \rho > 2 \), \( \lambda > 0 \), and let \( K \), \( \omega \), and \( \mu \) be as in Theorem 3.0.1. For \( 1 < p < \infty \), there exist constants \( C_1 \) and \( C_2 \) depending on \( \rho \), \( n \), \( d \), \( K \), and Lip\((A) \) (and on \( p \) for the case of \( C_1 \)) such that

\[
\left\| \left((N_\lambda \circ \mathcal{T}_\omega^\mu)f\right)^{1/p}\right\|_{L^p(\mu)} \leq \frac{C_1}{\lambda} \left\| f \right\|_{L^p(\mu)} \quad \text{and} \quad \mu(\{x \in \Gamma : (N_\lambda \circ \mathcal{T}_\omega^\mu)(f)(x) > m\}) \leq \frac{C_2}{\lambda \rho^{1/p}} \|f\|_{L^1(\mu)}.
\]

Trivially, \((N^b_a \circ \mathcal{T}_\omega^\mu)(f) \leq (N_{b-a} \circ \mathcal{T}_\omega^\mu)(f)\), thus Theorem 3.0.8 also holds replacing \( \lambda \) by \( b-a \) and \( N_\lambda \) by \( N^b_a \). In [JSW] it is shown that the results of Theorem 3.0.8 still hold when \( \rho = 2 \) for the particular case of the Hilbert transform. We do not pursue this endpoint result.

On the other hand, \( \mathcal{V}_\rho \circ \mathcal{T}_\omega^\mu \) is related to an important open problem posed by G. David and S. Semmes. We need some definitions to state it.

Recall that \( \mu \) is said to be \( n \)-dimensional Ahlfors-David regular, or simply AD regular, if there exists some constant \( C \) such that \( C^{-1} r^n \leq \mu(B(x, r)) \leq C r^n \) for all \( x \in \text{supp} \mu \) and \( 0 < r \leq \text{diam}(\text{supp} \mu) \). It is not difficult to see that such a measure \( \mu \) must be of the form \( \mu = h \mathcal{H}^n_{\text{supp} \mu} \), where \( h \) is some positive function bounded above and away from zero. A Borel set \( E \subset \mathbb{R}^d \) is called AD regular if the measure \( \mathcal{H}^n_E \) is AD regular. One says that \( \mu \) is uniformly \( n \)-rectifiable, or simply uniformly rectifiable, if there exist \( \theta, M > 0 \) so that, for each \( x \in \text{supp} \mu \) and \( R > 0 \), there is a Lipschitz mapping \( g \) from the \( n \)-dimensional ball
$B^n(0, R) \subset \mathbb{R}^n$ into $\mathbb{R}^d$ such that $\text{Lip}(g) \leq M$ and $\mu(B(x, R) \cap g(B^n(0, R))) \geq \theta R^n$, where $\text{Lip}(g)$ stands for the Lipschitz constant of $g$. In the language of [DS2], this means that $\text{supp} \mu$ has big pieces of Lipschitz images of $\mathbb{R}^n$. A Borel set $E \subset \mathbb{R}^d$ is called uniformly $n$-rectifiable if $\mathcal{H}^n_E$ is uniformly $n$-rectifiable. Of course, the $n$-dimensional graph of a Lipschitz function is uniformly $n$-rectifiable.

G. David and S. Semmes asked the following question, which is still open (see, for example, [Pa, Chapter 7]):

**Problem 3.0.9.** Is it true that an $n$-dimensional AD regular measure $\mu$ is uniformly $n$-rectifiable if and only if $R^n_\mu$ is bounded in $L^2(\mu)$?

It is proved in [DS1] that if $\mu$ is uniformly rectifiable, then $R^n_\mu$ is bounded in $L^2(\mu)$. However, the converse implication has been proved only in the case $n = 1$ and $d = 2$, by P. Mattila, M. Melnikov and J. Verdera [MMV], using the notion of curvature of measures (which seems to be useful only in this case).

Let $K(x)$ denote the $n$-dimensional Riesz kernel $x/|x|^{n+1}$ ($x \in \mathbb{R}^d$), so $R^n_\mu = T^n_\mu$. Combining some techniques from [DS2] and [To11], we will show that the $L^2$ boundedness of $V_\rho \circ T^n_\mu$ implies that $\mu$ is uniformly rectifiable (see Chapter 5 for more details). Moreover, we will also show that $V_\rho \circ T^n_\mu$ is bounded in $L^2(\mu)$ when $\mu$ is AD regular and uniformly rectifiable, thus we will obtain the following interesting result: *An $n$-dimensional AD regular measure $\mu$ is uniformly $n$-rectifiable if and only if $V_\rho \circ T^n_\mu$ is a bounded operator in $L^2(\mu)$* (see Corollary 5.0.10 in Chapter 5). This statement can be considered as a partial solution of Problem 3.0.9.

This chapter is organized as follows. In Section 3.1 we state some notation, definitions and preliminary results. In Section 3.2 we sketch the proof of our Main Theorem 3.0.1, and in the subsequent sections we give the detailed proof.

### 3.1 Preliminaries

As we said in the introduction, throughout all the chapter, $n$ and $d$ are two fixed integers such that $0 < n < d$. Given a point $x = (x^1, \ldots, x^d) \in \mathbb{R}^d$, we use the notation $\tilde{x} := (x^1, \ldots, x^n) \in \mathbb{R}^n$. Given a function $f : \mathbb{R}^m \to \mathbb{R}$, we denote by $\nabla f$ its gradient (when it makes sense), and by $\nabla^2 f$ the matrix of second derivatives of $f$. If $f$ depends on different points $x_1, x_2, \ldots \in \mathbb{R}^m$, then $\nabla_{x_i} f$ denotes the gradient of $f$ with respect to the $x_i$ variable, and analogously for $\nabla^2_{x_i} f$.

For two sets $F_1, F_2 \subset \mathbb{R}^d$, we denote by $\text{dist}_H(F_1, F_2)$ the Hausdorff distance between $F_1$ and $F_2$. We denote by $\mathcal{L}^n$ the Lebesgue measure on $\mathbb{R}^n$, and for the sake of simplicity, we set $\| \cdot \|_p := \| \cdot \|_{L^n(\mathcal{L}^n)}$ for $1 \leq p \leq \infty$, and $dy := d\mathcal{L}^n(y)$ for $y \in \mathbb{R}^n$. 
3. Variation for singular integrals on Lipschitz graphs: Smooth truncation

In this chapter, when we refer to the angle between two affine \(n\)-planes in \(\mathbb{R}^d\), we mean the angle between the \(n\)-dimensional subspaces associated to the \(n\)-planes. As usual, the letter ‘\(C\)’ stands for some constant which may change its value at different occurrences, and which quite often only depends on \(n\) and \(d\). The notation \(A \lesssim B (A \gtrsim B)\) means that there is some fixed constant \(C\) such that \(A \leq CB (A \geq CB)\), with \(C\) as above. Also, \(A \approx B\) is equivalent to \(A \lesssim B \lesssim A\).

3.1.1 More about the families of truncations \(\omega\)

Given \(x \in \mathbb{R}^d\), \(0 < \epsilon \leq \delta\), and a finite Borel measure \(\mu\), we set \(\omega^\delta(x) := \omega(x) - \omega_\delta(x)\) and we define

\[(K\omega^\delta * \mu)(x) := \int \omega^\delta(y)K(x-y)\,d\mu(y),\]

thus \((K\omega^\delta * \mu)(x) = (K\omega_* \mu)(x) - (K\omega_\delta * \mu)(x)\).

For \(m \in \mathbb{N}\), \(x \in \mathbb{R}^m\), and \(R \geq r > 0\), we denote by \(B^m(x,r)\) the closed ball of \(\mathbb{R}^m\) with center \(x\) and radius \(r\), and by \(A^m(x,r,R)\) the closed annulus of \(\mathbb{R}^m\) centered at \(x\) with inner radius \(r\) and outer radius \(R\). We also use the notation \(B(x,r)\) and \(A(x,r,R)\) when there is no possible confusion about \(m\).

If we take \(\omega = \varphi\), then \(\varphi^\delta = \varphi - \varphi_\delta\) is non negative, and \(\text{supp} \varphi^\delta \subset A^d(0,2.1\epsilon \sqrt{n},3\delta \sqrt{n})\). Moreover, \(\sum_{j \in \mathbb{Z}} \varphi^{2^{-j-1}}(x) = 1\) for \(x \neq 0\), and there are at most two terms that do not vanish in the previous sum for a given \(x \in \mathbb{R}^d\). For the case of \(\omega = \tilde{\varphi}\), one also has \(\text{supp} \tilde{\varphi}^\delta \subset A^n(0,2.1\epsilon \sqrt{n},3\delta \sqrt{n}) \times \mathbb{R}^{d-n} \subset \mathbb{R}^d\) and \(\sum_{j \in \mathbb{Z}} \tilde{\varphi}^{2^{-j-1}}(x) = 1\) for \(x \neq 0\).

3.1.2 The \(\alpha\) and \(\beta\) coefficients. Special dyadic lattice

Given \(m \in \mathbb{N}\), \(\lambda > 0\), and a cube \(Q \subset \mathbb{R}^m\) (i.e. \(Q := [0,b)^m + a\) with \(a \in \mathbb{R}^m\) and \(b > 0\)), \(\ell(Q)\) denotes the side length of \(Q\), \(z_Q\) denotes the center of \(Q\) and \(\lambda Q\) denotes the cube with center \(z_Q\) and side length \(\lambda \ell(Q)\). Throughout the chapter, we will only use cubes with sides parallel to the axes.

Let \(\mu\) be a locally finite Borel measure on \(\mathbb{R}^d\). Given \(1 \leq p < \infty\) and a cube \(Q \subset \mathbb{R}^d\), one sets (see [DS2])

\[
\beta_{p,\mu}(Q) = \inf_L \left\{ \frac{1}{\ell(Q)^n} \int_{2Q} \left( \frac{\text{dist}(y,L)}{\ell(Q)} \right)^p \, d\mu(y) \right\}^{1/p},
\]

where the infimum is taken over all \(n\)-planes \(L\) in \(\mathbb{R}^d\). For \(p = \infty\) one replaces the \(L^p\) norm by the supremum norm:

\[
\beta_{\infty,\mu}(Q) = \inf_L \left\{ \sup_{y \in \text{supp} \mu \cap 2Q} \frac{\text{dist}(y,L)}{\ell(Q)} \right\},
\]
where the infimum is taken over all $n$-planes $L$ in $\mathbb{R}^d$ again. These coefficients were introduced by P. W. Jones in [Jn1] for $p = \infty$ and by G. David and S. Semmes in [DS1] for $1 \leq p < \infty$.

Let $F \subset \mathbb{R}^d$ be the closure of an open set. Given two finite Borel measures $\sigma, \nu$ on $\mathbb{R}^d$, one sets
\[
\text{dist}_F(\sigma, \nu) := \sup \left\{ \left| \int f \, d\sigma - \int f \, d\nu \right| : \text{Lip}(f) \leq 1, \text{supp} f \subset F \right\}. \tag{3.1.3}
\]
It is easy to check that this is a distance in the space of finite Borel measures $\sigma$ such that $\text{supp} \sigma \subset F$ and $\sigma(\partial F) = 0$. Moreover, it turns out that this distance is a variant of the well known Wasserstein distance $W_1$ from optimal transportation (see [Vi, Chapter 1]). See [Ma1, Chapter 14] for other properties of $\text{dist}_F$.

Given a cube $Q$ which intersects $\text{supp} \mu$, consider the closed ball $B_Q := B(z_Q, 6\sqrt{d} \ell(Q))$. Then one defines (see [To11])
\[
\alpha^n_\mu(Q) := \frac{1}{\ell(Q)^{n+1}} \inf_{c \geq 0, L} \text{dist}_{B_Q}(\mu, cH^n_L), \tag{3.1.4}
\]
where the infimum is taken over all constants $c \geq 0$ and all $n$-planes $L$ in $\mathbb{R}^d$. For convenience, if $Q$ does not intersect $\text{supp} \mu$, we set $\alpha^n_\mu(Q) = 0$. To simplify notation, sometimes we will write $\alpha_\mu(Q)$ or $\alpha(Q)$ instead of $\alpha^n_\mu(Q)$ (and analogously for the $\beta$’s).

The following result characterizes uniform rectifiability in terms of the $\alpha$ and $\beta$ coefficients.

**Theorem 3.1.1.** Let $\mu$ be an $n$-dimensional AD regular measure on $\mathbb{R}^d$, and consider any $p \in [1, 2]$. Then, the following are equivalent:

(a) $\mu$ is uniformly $n$-rectifiable.

(b) For any cube $R \subset \mathbb{R}^d$,
\[
\sum_{Q \in D_{Rd}(R)} \beta_{p,\mu}(Q)^2 \ell(Q)^n \leq C \ell(R)^n \tag{3.1.5}
\]
with $C$ independent of $R$, where $D_{Rd}(R)$ stands for the collection of cubes of $\mathbb{R}^d$ contained in $R$ which are obtained by splitting $R$ dyadically.

(c) There exists $C > 0$ such that, for any cube $R \subset \mathbb{R}^d$,
\[
\sum_{Q \in D_{Rd}(R)} \alpha_\mu(Q)^2 \ell(Q)^n \leq C \ell(R)^n. \tag{3.1.6}
\]

The equivalence $(a) \iff (b)$ in Theorem 3.1.1 was proved by G. David and S. Semmes in [DS1], and the equivalence $(a) \iff (c)$ was proved by X. Tolsa in [To11].

In this chapter we will use a slightly different definition of the $\alpha$ and $\beta$ coefficients adapted to the $n$-uniformly rectifiable measure $\mu = fH^n_{\Gamma}$, where $\Gamma := \{ x \in \mathbb{R}^d : x = (\tilde{x}, A(\tilde{x})) \}$ is the $n$-dimensional graph of a given Lipschitz function $A : \mathbb{R}^n \to \mathbb{R}^{d-n}$ and $f \in L^\infty(H^n_{\Gamma})$ satisfies $f(x) \approx 1$ for almost all $x \in \Gamma$. To this end, we need to introduce a special dyadic lattice of sets related to $\Gamma$. Given a cube $\tilde{Q} \subset \mathbb{R}^n$ (i.e. $\tilde{Q} := [0, b)^n + a$ with $a \in \mathbb{R}^n$ and $b > 0$), we define
3. Variation for singular integrals on Lipschitz graphs: Smooth truncation

Let \( D \) denote the standard dyadic lattice of \( \mathbb{R}^n \), and set \( \mathcal{D} := \{ Q : \bar{Q} \in \mathcal{D} \} \). It is easy to check that the v-cubes of \( \mathcal{D} \) intersected with \( \Gamma \) provide a dyadic lattice associated to the graph \( \Gamma \) in the sense of [Da1, Appendix 1]. Finally, for \( m \in \mathbb{Z} \), set \( \mathcal{D}_m := \{ Q \in \mathcal{D} : \ell(Q) = 2^{-m} \} \).

Fix a constant \( C_\Gamma > 10 \sqrt{n}(1 + \text{Lip}(A)) \) (the precise value of \( C_\Gamma \) will not be relevant in the proofs given in this chapter, it only has to be “big enough”). Given \( 1 \leq p \leq \infty \) and a v-cube \( Q \subset \mathbb{R}^d \), we define the coefficient \( \beta_{p,\mu}(Q) \) as in (3.1.1) and (3.1.2) but replacing \( 2Q \) by \( C_\Gamma Q \). We also define \( \alpha_\mu(Q) \) as in (3.1.4) but taking \( B_Q := B(\bar{Q}, C_\Gamma \ell(Q)) \times \mathbb{R}^{d-n} \subset \mathbb{R}^d \). This new definition of the \( \alpha \) and \( \beta \) coefficients (adapted to the graph \( \Gamma \)) is the one that we will use in the whole chapter.

**Remark 3.1.2.** It is an exercise to check that, with this new definition of the \( \alpha \)'s and \( \beta \)'s, inequalities (3.1.5) and (3.1.6) of Theorem 3.1.1 still hold. Moreover, the following is an easy consequence of (3.1.5) and (3.1.6): Let \( \Gamma \) be an \( n \)-dimensional Lipschitz graph, \( f \in L^p(\mathcal{H}^p_\Gamma) \) such that \( C^{-1} \leq f(x) \leq C \) for almost all \( x \in \Gamma \), and \( \mu = f\mathcal{H}^p_\Gamma \). Let \( 1 \leq p \leq 2 \). Given \( C_1, C_2, C_3 \geq 1 \), there exists a constant \( C_4 > 0 \) such that, for any \( R \in \mathcal{D} \),

\[
\sum_{Q \in \mathcal{D} : Q \subset C_1 R} \left( \beta_{p,\mu}(C_2 Q)^2 + \alpha_\mu(C_3 Q)^2 \right) \mu(Q) \leq C_4 \mu(R),
\]

and the dependence of \( C_4 \) with respect to \( \Gamma \) is only on \( \text{Lip}(A) \).

**Remark 3.1.3.** It is shown in [To11, Lemma 3.2], that \( \beta_{1,\mu}(Q) \preceq \alpha_\mu(Q) \) for all \( Q \in \mathcal{D} \). Given \( Q \in \mathcal{D} \), let \( L_Q \) be a minimizing \( n \)-plane for \( \alpha_\mu(Q) \). In general, we don’t have \( \beta_{\infty,\mu}(Q) \preceq \beta_{1,\mu}(Q) \), so given \( x \in \text{supp} \mu \cap C_\Gamma Q \), we cannot ensure that \( \text{dist}(x, L_Q) \preceq \alpha_\mu(Q) \). But it is shown in [To11, Lemma 5.2] that \( \text{dist}(x, L_Q) \preceq \sum_{R \in \mathcal{D} : x \in R \subset Q} \alpha_\mu(R) \ell(R) \), and in particular, if \( P \in \mathcal{D} \) is such that \( P \subset Q \) and \( x \in \text{supp} \mu \cap C_\Gamma P \), and \( L_P \) denotes a minimizing \( n \)-plane for \( \alpha_\mu(P) \), one has (see [To11, Remark 5.3])

\[
\text{dist}(x, L_Q) \preceq \text{dist}(x, L_P) + \sum_{R \in \mathcal{D} : P \subset R \subset Q} \alpha_\mu(R) \ell(R).
\]

3.1.3 Martingales

First of all, let us recall a particular case of Lépingle’s inequality (see [JSW], or [Lé] and [JKRW, Theorem 6.4] for martingales in a probability space):

**Theorem 3.1.4.** Let \( (X, \Sigma, \lambda) \) be a \( \sigma \)-finite measure space and \( \rho > 2 \). Then, there exist constants \( C_1, C_2 > 0 \) such that, for every martingale \( \mathcal{G} := \{ G_m \}_{m \in \mathbb{Z}} \in L^2(\lambda) \),

\[
\| \mathcal{V}_\rho(\mathcal{G}) \|_{L^2(\lambda)} \leq C_1 \| \mathcal{G} \|_{L^2(\lambda)} \quad \text{and} \quad \| \mathcal{O}(\mathcal{G}) \|_{L^2(\lambda)} \leq C_2 \| \mathcal{G} \|_{L^2(\lambda)},
\]
where \( \|G\|_{L^2(\lambda)} := \sup_{m \in \mathbb{Z}} \|G_m\|_{L^2(\lambda)} \). The constants \( C_1 \) and \( C_2 \) do not depend on the measure \( \lambda \), and \( C_2 \) neither depends on the fixed sequence that defines \( \mathcal{O} \).

To prove Theorem 3.0.1, we need to introduce a particular martingale, and to review some known results.

**Lemma 3.1.5.** Fix a cube \( \widetilde{P} \subset \mathbb{R}^n \) (not necessarily dyadic) and a Lipschitz graph \( \Gamma := \{ x \in \mathbb{R}^d : x = (\tilde{x}, A(\tilde{x})) \} \) such that \( \text{supp} A \subset \widetilde{P} \). Consider the measure \( \mu := f \mathcal{H}^n_{L} \), where \( f(x) = 1 \) for all \( \tilde{x} \in \widetilde{P}^c \) and \( C_0^{-1} \leq f(x) \leq C_0 \) for some fixed constant \( C_0 > 0 \). Also set \( P := \widetilde{P} \times \mathbb{R}^{d-n} \). Then, the following hold:

\[
T_* \mu \in L^1_{loc}(\mu), \quad T_*(\chi_E \mu) \in L^1_{loc}(\mu) \quad \text{for every compact set } E \subset \mathbb{R}^d, \quad \|T\mu\|_{L^2(\mu)} \lesssim \mu(P)^{1/2}. \tag{3.1.8}
\]

**Remark 3.1.6.** To avoid the problem of non-integrability near infinity, for this type of measures \( \mu \) we redefine \( T_* \mu(x) := \lim_{M \to \infty} (K\chi_{M} \ast \mu)(x) \), which exists because \( \mu \) is flat outside a compact set and \( K \) is odd. All the results in this chapter remain valid with this new definition and the adjustments that have to be done in the proofs are minimal. In this chapter, we will deal with other integrals which concern the kernel \( K \) and the measure \( \mu \) near infinity. The non-integrability problem can be avoided in the same manner.

**Proof of Lemma 3.1.5.** It is known that the operator \( T_*^\mu \) is bounded in \( L^2(\mu) \), because \( T_*^\mu \) is the maximal operator associated to a Calderón-Zygmund singular integral and \( \mu \) is an uniformly rectifiable measure (see [DS1]). Thus, \( T_*(\chi_E \mu) = T_*^\mu(\chi_E) \in L^1_{loc}(\mu) \) for every compact set \( E \subset \mathbb{R}^d \).

We are going to check that \( \|T_* \mu\|_{L^2(\mu)} \lesssim \mu(P)^{1/2} \). This will imply that \( T_* \mu \in L^1_{loc}(\mu) \) and, since \( T_* \mu \) exists (because \( \mu \) is uniformly rectifiable) and \( |T_* \mu| \leq T_* \mu \), we will also obtain \( \|T_* \mu\|_{L^2(\mu)} \lesssim \mu(P)^{1/2} \); so the lemma will be proved.

Using that \( T_*^\mu \) is bounded in \( L^2(\mu) \), we have

\[
\|T_* \mu\|_{L^2(\mu)} \leq \|T_*(\chi_{3P} \mu)\|_{L^2(\mu)} + \|T_*(\chi_{(3P)^c} \mu)\|_{L^2(\mu)} \lesssim \mu(P)^{1/2} + \|T_*(\chi_{(3P)^c} \mu)\|_{L^2(\mu)}. \tag{3.1.10}
\]

Set \( L := \mathbb{R}^n \times \{0\}^{d-n} \subset \mathbb{R}^d \); obviously \( \chi_{3P} = \mathcal{H}^n_{L \setminus P} \). Since \( L \) is an n-plane and \( K \) is odd, \( T_* \mathcal{H}^n_{L}(x) = 0 \) for all \( x \in L \). Thus,

\[
\|T_* \mathcal{H}^n_{L \setminus 3P}\|_{L^2(\mathcal{H}^n_{L \setminus 3P})} \leq \|T_* \mathcal{H}^n_{L\setminus P}\|_{L^2(\mathcal{H}^n_{L \setminus P})} \lesssim \mu(P)^{1/2}. \tag{3.1.11}
\]

Set \( \tilde{z}_P := (\tilde{z}_P, 0, \ldots, 0) \in L \) (recall that \( \tilde{z}_P \) denotes the center of \( \widetilde{P} \)). It is obvious that \( \int \chi_{\epsilon}(z_P - y)K(z_P - y) d\mathcal{H}^n_{L \setminus 3P}(y) = 0 \) for all \( \epsilon > 0 \). Thus, given \( \epsilon \in \text{supp} \mu \cap P \),

\[
\left| (K\chi_{\epsilon} \ast \mathcal{H}^n_{L \setminus 3P})(x) \right| \leq \int \chi_{\epsilon}(x - y)|K(x - y) - K(z_P - y)| d\mathcal{H}^n_{L \setminus 3P}(y) + \int \left| \chi_{\epsilon}(x - y) - \chi_{\epsilon}(z_P - y) \right||K(z_P - y)| d\mathcal{H}^n_{L \setminus 3P}(y).
\]
Since \( \Gamma \) is a Lipschitz graph, \( |x - z_p| \leq \ell(P) \). So, the first term on right hand side of the previous inequality is easily bounded by an absolute constant independent of \( \epsilon \), by standard arguments. For the second term, notice that \( \text{supp}(\chi_{\epsilon}(x - \cdot) - \chi_{\epsilon}(z_p - \cdot)) \cap (L \setminus 3P) = \emptyset \) for all \( \epsilon < \ell(P) \), and \( \mathcal{H}^n_\epsilon \{ (y \in \mathbb{R}^n : \chi_{\epsilon}(x - y) - \chi_{\epsilon}(z_p - y) \neq 0) \} \lesssim \ell(P)e^{n-1} \) for all \( \epsilon \geq \ell(P) \). Therefore, since \( |z_p - y| \approx \epsilon \) for all \( y \in \text{supp}(\chi_{\epsilon}(x - \cdot) - \chi_{\epsilon}(z_p - \cdot)) \cap (L \setminus 3P) \), the second term can also be estimated by an absolute constant. Thus, we conclude \( T_\epsilon \mathcal{H}^n_{L \setminus 3P}(x) = \sup_{\epsilon > 0} |(K\chi_{\epsilon} \ast \mathcal{H}^n_{L \setminus 3P})(x)| \lesssim 1 \) for all \( x \in \text{supp}\mu \cap P \).

Using the previous observations and (3.1.11), we have
\[
\|T_\epsilon(\chi(3P)^c\mu)\|_{L^2(\mu)}^2 = \|T_\epsilon \mathcal{H}^n_{L \setminus 3P}\|_{L^2(\chi_{3P}\mu)}^2 + \|T_\epsilon \mathcal{H}^n_{L \setminus 3P}\|_{L^2(\chi_{3P}\mu)}^2 \\
\leq \|T_\epsilon \mathcal{H}^n_{L \setminus 3P}\|_{L^2(\chi_{3P}\mu)}^2 + \|T_\epsilon \mathcal{H}^n_{L \setminus 3P}\|_{L^2(\chi_{3P}\mu)}^2 \lesssim \mu(P),
\]
which, combined with (3.1.10), gives \( \|T_\epsilon \mu\|_{L^2(\mu)} \lesssim \mu(P)^{1/2} \), as desired. \( \square \)

We are ready to define the martingale. Let \( P \) and \( \mu \) be as in Lemma 3.1.5. Given \( m \in \mathbb{Z} \) and \( a \in \mathbb{R}^n \), we set
\[
\tilde{D}_m^a := a + [0, 2^{-m})^n \subset \mathbb{R}^n \quad \text{and} \quad D_m^a := \tilde{D}_m^a \times \mathbb{R}^{-d-n} \subset \mathbb{R}^d.
\]
Set \( \mathcal{D}_m^a := \{ D_m^{a+2^{-m}k} \subset \mathbb{R}^d : k \in \mathbb{Z}^n \} \) (notice that \( \mathcal{D}_m^a \) coincides with \( \mathcal{D}_m \) translated by a parameter \( a \in \mathbb{R}^n \) and, for a fixed \( a \), \( \bigcup_{m \in \mathbb{Z}} \mathcal{D}_m^a \) is a translation of the standard dyadic lattice). Notice that \( \mu(D_m^a) \approx 2^{-mn} \) for all \( m \in \mathbb{Z}, a \in \mathbb{R}^n \). For \( D \in \mathcal{D}_m^a \) and \( x \in D \), we set
\[
E_D \mu(x) := \frac{1}{\mu(D)} \int_D \int_{D^c} K(z - y) \, d\mu(y) \, d\mu(z)
\]
(take into account Remark 3.1.6 for the meaning of \( \int_{D^c} K(z - y) \, d\mu(y) \)). Finally, for \( x \in \mathbb{R}^d \), we define the martingale \( E_m^a \mu(x) := \sum_{D \in \mathcal{D}_m} \chi_D(x) E_D \mu(x), m \in \mathbb{Z} \).

Let us make some comments to understand better the nature of \( E_m^a \mu \). First of all notice that, since \( \mu(\partial D) = 0 \), for any \( D \in \mathcal{D}_m^a \) and \( \mu \)-almost all \( z \in D \) we have
\[
\int_{D^c} K(z - y) \, d\mu(y) = \lim_{\epsilon \to 0} \int_{D^c} \chi_{\epsilon}(z - y) K(z - y) \, d\mu(y),
\]
and for any \( \epsilon > 0 \), we have
\[
\int_D \int_{D^c} \chi_{\epsilon}(z - y) K(z - y) \, d\mu(y) \, d\mu(z) = 0
\]
because of the antisymmetry of \( K \). Therefore, by (3.1.12), (3.1.13), (3.8), and the dominated convergence theorem, \( \int_D |\int_{D^c} K(z - y) \, d\mu(y)| \, d\mu(z) < \infty \) (in particular, we have seen that \( E_m^a \mu \) is well defined) and \( \int_D T(\chi_D \mu) \, d\mu = 0 \). Using this and (3.1.12), we finally have that
\[
E_m^a \mu(x) = \frac{1}{\mu(D)} \int_D T(\chi_D \mu) \, d\mu = \frac{1}{\mu(D)} \int_D T \mu \, d\mu
\]
for \( x \in D \in \mathcal{D}_m^a \), thus \( E^a_m \mu(x) \) is the average of the function \( T \mu \) on the \( \nu \)-cube \( D \in \mathcal{D}_m^a \) which contains \( x \). So, it is completely clear that, for a fixed \( a \in \mathbb{R}^n \), \( \{E^a_m \mu\}_{m \in \mathbb{Z}} \) is a martingale. In [MV] it is shown that \( \{E^a_m \mu\}_{m \in \mathbb{Z}} \) is well defined and it is a martingale without the assumption of the existence of \( T \mu \) (i.e., for more general measures \( \mu \)).

Now, we can use (3.1.14), the \( L^2 \) boundedness of the dyadic maximal operator and (3.1.9) to deduce that

\[
\|E^a_m \mu\|_{L^2(\mu)} \lesssim \|T \mu\|_{L^2(\mu)} \lesssim \mu(P)^{1/2}
\]

(3.1.15) for all \( a \in \mathbb{R}^n \) and \( m \in \mathbb{Z} \), where the constants that appear in the previous inequalities only depend on \( C_0, n, d \) and \( \text{Lip}(A) \).

Set \( E^a \mu := \{E^a_m \mu\}_{m \in \mathbb{Z}} \). Then, the martingale \( E^a \mu \) belongs to \( L^2(\mu) \) by (3.1.15); thus by Theorem 3.1.4, for all \( a \in \mathbb{R}^n \),

\[
\|\mathcal{V}_\rho (E^a \mu)\|_{L^2(\mu)} \lesssim \|E^a \mu\|_{L^2(\mu)} \lesssim \mu(P)^{1/2} \quad \text{for } \rho > 2,
\]

(3.1.16)

where the constants in the previous inequalities only depend on \( C_0, n, d \) and \( \text{Lip}(A) \) (and on \( \rho \), in the case of \( \mathcal{V}_\rho \)).

Finally, for \( x \in \mathbb{R}^d \), we define

\[
E_m \mu(x) := 2^{mn} \int_{\{a: x \in D_m^a\}} E^a_m \mu(x) \, da
\]

(notice that \( \mathcal{L}^n(\{a: x \in D_m^a\}) = 2^{-mn} \)). Thus, \( E_m \mu \) is an average (of the \( m \)’th term) of some martingales depending on a parameter \( a \in \mathbb{R}^n \).

Set \( E \mu := \{E_m \mu\}_{m \in \mathbb{Z}} \). We want to obtain estimates like (3.1.16) for \( \mathcal{V}_\rho (E \mu) \) and \( \mathcal{O}(E \mu) \). We will only show the details for \( \mathcal{V}_\rho (E \mu) \), because the case of \( \mathcal{O}(E \mu) \) follows by similar arguments.

One can easily check that \( E_m \mu(x) = 2^{Mn} \int_{[0,2^{-M}]^n} E^a_m \mu(x) \, da \) for all \( m, M \in \mathbb{Z} \) with \( M \leq m \). Therefore, for all \( M, r, s \in \mathbb{Z} \) with \( M \leq r \leq s \), we have

\[
E_r \mu(x) - E_s \mu(x) = 2^{Mn} \int_{[0,2^{-M}]^n} (E^a_r \mu(x) - E^a_s \mu(x)) \, da.
\]

(3.1.17)

Given \( M \in \mathbb{Z} \), we consider the auxiliary transformation

\[
\mathcal{V}_{\rho, M}(E \mu)(x) := \sup_{\{r_m\}} \left( \sum_{m \in \mathbb{Z}} |E_{r_m+1} \mu(x) - E_{r_m} \mu(x)|^\rho \right)^{1/\rho},
\]

where the pointwise supremum is taken over all decreasing sequences of integers \( \{r_m\}_{m \in \mathbb{Z}} \) such that \( r_m \geq M \) for all \( m \in \mathbb{Z} \). With this definition it is obvious that the sequence
\( \{ V_{p,M}(E\mu)(x) \}_{M \in \mathbb{Z}} \) is non increasing and \( V_{p}(E\mu)(x) = \lim_{M \to -\infty} V_{p,M}(E\mu)(x) \) for all \( x \in \mathbb{R}^d \). Minkowski’s integral inequality and (3.1.17) yield the pointwise estimate

\[
V_{p,M}(E\mu)(x) = \sup_{\{r_m : r_m \geq M\}} \left( \sum_{m \in \mathbb{Z}} |E_{r_{m+1}} \mu(x) - E_{r_m} \mu(x)|^p \right)^{1/p} \\
\leq 2^{Mn} \int_{[0,2^{-n}]} \sup_{\{r_m\}} \left( \sum_{m \in \mathbb{Z}} |E_{\tilde{r}_{m+1}} \mu(x) - E_{\tilde{r}_m} \mu(x)|^p \right)^{1/p} da \\
= 2^{Mn} \int_{[0,2^{-n}]} V_{p}(E^a \mu)(x) da.
\]

Therefore, by the previous estimate, Minkowski’s integral inequality and (3.1.16),

\[
\| V_{p,M}(E\mu) \|_{L^2(\mu)} \leq 2^{Mn} \int_{[0,2^{-n}]} \| V_{p}(E^a \mu) \|_{L^2(\mu)} da \leq C \mu(P)^{1/2},
\]

where \( C > 0 \) only depends on \( C_0, n, d, \text{Lip}(A) \), and \( \rho \). By the monotone convergence theorem, we conclude that \( \| V_{p}(E\mu) \|_{L^2(\mu)} \leq \mu(P)^{1/2} \). Thus we have proved the following theorem (which can be considered the starting point to prove Main Theorem 3.0.1):

**Theorem 3.1.7.** Fix a cube \( \tilde{P} \subset \mathbb{R}^n \). Set \( \Gamma := \{ x \in \mathbb{R}^d : x = (\tilde{x}, A(\tilde{x})) \} \), where \( A : \mathbb{R}^n \to \mathbb{R}^{d-n} \) is a Lipschitz function supported in \( \tilde{P} \), and set \( P := \tilde{P} \times \mathbb{R}^{d-n} \). Set \( \mu := f \mathcal{H}^n_1 \), where \( f(x) = 1 \) for all \( \tilde{x} \in \tilde{P}^c \) and \( C_0^{-1} \leq f(x) \leq C_0 \) for all \( \tilde{x} \in \tilde{P} \), for some constant \( C_0 > 0 \).

Let \( \rho > 2 \). Then, there exist constants \( C_1, C_2 > 0 \) such that

\[
\| V_{p}(E\mu) \|_{L^2(\mu)} \leq C_1 \mu(P)^{1/2} \quad \text{and} \quad \| O(\mu) \|_{L^2(\mu)} \leq C_2 \mu(P)^{1/2},
\]

where \( C_1 \) and \( C_2 \) only depend on \( C_0, n, d, \text{Lip}(A) \) (and on \( \rho \) in the case of \( C_1 \)).

We need to introduce additional notation in order to express \( E_m \mu \) in a more convenient way for our purposes. Let \( \mu_1, \ldots, \mu_k \) be a finite collection of positive Borel measures such that \( \mu_l(D_{m}^a) > 0 \) for all \( a \in \mathbb{R}^n, m \in \mathbb{Z} \) and \( l = 1, \ldots, k \). Given \( m \in \mathbb{Z} \) and \( x_1, \ldots, x_i, y_1, \ldots, y_j \in \mathbb{R}^d \), we define

\[
\Lambda^1_{m_1} \cdots \Lambda^k_{m_k}(x_1, \ldots, x_i; y_1, \ldots, y_j) := 2^{mn} \int_{\{ a : x_i \in D_{m}^a \cap y_j \in D_{m}^a \}} \frac{da}{\prod_{l=1}^k \mu_l(D_{m}^a)}.
\]

Then, by Fubini’s theorem,

\[
E_m \mu(x) = \int_{\{ a : x \in D_{m}^a \}} \frac{2^{mn} \mu(D_{m}^a)}{\mu(D_{m}^a)} \int_{D_{m}^a} \int_{D_{m}^a} K(z - y) d\mu(y) d\mu(z) da \\
= \int \left( 2^{mn} \int_{\{ a : x,z \in D_{m}^a, y \notin D_{m}^a \}} \frac{da}{\mu(D_{m}^a)} \right) K(z - y) d\mu(z) d\mu(y) \\
= \int \Lambda^1_{m}(x, z; y) K(z - y) d\mu(z) d\mu(y).
\]
3.2 Sketch of the proof of Main Theorem 3.0.1

The proof relies on two basic facts: the known $L^2$ boundedness of the $\rho$-variation and oscillation of martingales explained in the previous section and the good geometric properties of Lipschitz graphs from a measure-theoretic point of view.

As we said above, the starting point of the proof is Theorem 3.1.7, where the $L^2$ boundedness of the $\rho$-variation and oscillation (of a convex combination) of some particular martingales is stated. So, the first step consists in relating the results on martingales in Theorem 3.1.7 with the $\rho$-variation and oscillation of singular integrals on Lipschitz graphs, and this is the aim of the following two theorems:

**Theorem 3.2.1.** Let $\Gamma$ and $\mu$ be as in Theorem 3.1.7. For each $x \in \Gamma$, define
\[ W_\mu(x)^2 := \sum_{m \in \mathbb{Z}} |(K_{\tilde{\phi}_{2^{-m}} \ast \mu})(x) - E_m \mu(x)|^2. \] (3.2.1)

Then, \[ \|W_\mu\|_{L^2(\mu)}^2 \leq C_1 \sum_{Q \in D} (\alpha_\mu(Q)^2 + \beta_2 \mu(Q)^2) \mu(Q), \] where $C_1, C_2 > 0$ depend only on $C_0, n, d, K,$ and Lip$(A)$.

**Theorem 3.2.2.** Let $\Gamma$ and $\mu$ be as in Theorem 3.1.7. For each $x \in \Gamma$, define
\[ S_\mu(x)^2 := \sup_{\{\epsilon_m\}} \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}; \epsilon_m, \epsilon_{m+1} \in I_j} |(K_{\tilde{\phi}_{\epsilon_m} \ast \mu})(x)|^2, \] (3.2.2)

where $I_j = [2^{-j-1}, 2^{-j})$ and the supremum is taken over all decreasing sequences of positive numbers $\{\epsilon_m\}_{m \in \mathbb{Z}}$. Then, \[ \|S_\mu\|_{L^2(\mu)}^2 \leq C \sum_{Q \in D} (\alpha_\mu(Q)^2 + \beta_2 \mu(Q)^2) \mu(Q), \] where $C > 0$ only depends on $C_0, n, d, K,$ and Lip$(A)$.

Two fundamental tools to study $W_\mu$ and $S_\mu$ are the $\alpha$ and $\beta$ coefficients, which will be used to measure the flatness of $\Gamma$ at different scales, in order to estimate the terms which appear in the sums in (3.2.1) and (3.2.2). This will be done in Sections 3.3 and 3.4. To use the $\alpha$ coefficients to relate the $\rho$-variation of martingales with the $\rho$-variation of singular integrals, it is a key fact that we are considering a family of smooth truncations like $\tilde{\phi}$, instead of $\chi$, because the $\alpha$’s are defined in terms of Lipschitz functions. Moreover, for the moment, we are taking a truncation only on the first $n$-coordinates (i.e., $\tilde{\phi}$ instead of $\phi$) because the average of martingales that we are using is taken over the parameter $a \in \mathbb{R}^n$, using the v-cubes $D^a_n$ (see Subsection 3.1.3).

Combining Theorem 3.2.1 and Theorem 3.2.2 with the $L^2$ estimates of the $\rho$-variation and oscillation on the average of martingales $E\mu$ in Theorem 3.1.7, we are able to obtain local $L^2$ estimates of $\mathcal{V}_\rho \circ T^{\mathcal{H}_\rho}_\bar{\varphi}$ and $\mathcal{O} \circ T^{\mathcal{H}_\rho}_\bar{\varphi}$ when $\Gamma$ is any Lipschitz graph. More precisely, we separate the sum in the definition of $\mathcal{V}_\rho \circ T^{\mathcal{H}_\rho}_\bar{\varphi}$ into two parts, which are classically called
short and long variation (and analogously for \( \mathcal{O} \circ \mathcal{T}^{H^p}_\varphi \)). The short variation corresponds to the sum \( S\mu \) in Theorem 3.2.2 (here \( \mu \) is a suitable modification of \( \mathcal{H}^p_\Gamma \)), where the indices run over \( m \in \mathbb{Z} \) such that both \( \epsilon_m \) and \( \epsilon_{m+1} \) lie in the same dyadic interval, and can be handled using the \( \alpha \)'s and \( \beta \)'s. The long variation corresponds to the sum over the indices \( m \in \mathbb{Z} \) such that \( \epsilon_m \) and \( \epsilon_{m+1} \) lie in different dyadic intervals, so one may assume that the \( \epsilon_m \)'s are dyadic numbers. It is handled by comparing \( K\tilde{\varphi}_{2^{-m}} \ast \mu \) with \( E_{m}\mu \), and then using Theorem 3.2.1 and the fact the \( \rho \)-variation and oscillation of \( E\mu \) are bounded in \( L^2(\mu) \), by Theorem 3.1.7. This will be done in Section 3.5 (see Theorem 3.5.1).

Using the local \( L^2 \) estimates of Theorem 3.5.1, combined with rather standard techniques in Calderón-Zygmund theory, in Section 3.6 we obtain the \( H^1(\mathcal{H}^p_\Gamma) \rightarrow L^1(\mathcal{H}^p_\Gamma) \) and \( L^\infty(\mathcal{H}^p_\Gamma) \rightarrow BMO(\mathcal{H}^p_\Gamma) \) boundedness of \( \mathcal{V}_\nu \circ \mathcal{T}^{H^p}_\varphi \) and \( \mathcal{O} \circ \mathcal{T}^{H^p}_\varphi \). Then, by interpolation, we obtain the \( L^p \) boundedness of these operators in the whole range \( 1 < p < \infty \), and in particular the \( L^2 \) boundedness (see Theorem 3.6.1).

The next step is to replace the family of smooth truncations \( \tilde{\varphi} \) by the other families of truncations \( \omega \). This will be done in Chapter 4. We focus our interest on the case \( \omega = \chi \), because we think that it is the most important one and the other (easier) cases follow using similar arguments. We obtain the \( L^2 \) boundedness of \( \mathcal{V}_\rho \circ \mathcal{T}^{H^p}_\chi \) and \( \mathcal{O} \circ \mathcal{T}^{H^p}_\chi \) by comparing these operators with \( \mathcal{V}_\rho \circ \mathcal{T}^{H^p}_\varphi \) and \( \mathcal{O} \circ \mathcal{T}^{H^p}_\varphi \), and by estimating the difference in terms of the \( \alpha \) and \( \beta \) coefficients, decomposing a function \( f \in L^2(\mathcal{H}^p_\Gamma) \) using a suitable wavelet basis. It is in this step where we need the assumption \( \text{Lip}(A) < 1 \) for \( \omega = \chi \), if \( \omega \in \{ \varphi, \tilde{\varphi} \} \) then \( \text{Lip}(A) < \infty \) suffices. This is done in Section 4.1 (see Theorem 4.1.1), in Chapter 4.

Finally, in Section 4.2 (see Theorem 4.2.1) we show that the \( L^2 \) boundedness of \( \mathcal{V}_\rho \circ \mathcal{T}^{H^p}_\chi \) and \( \mathcal{O} \circ \mathcal{T}^{H^p}_\chi \) yields the \( L^1(\mathcal{H}^p_\Gamma) \rightarrow L^{1,\infty}(\mathcal{H}^p_\Gamma) \) and \( L^\infty(\mathcal{H}^p_\Gamma) \rightarrow BMO(\mathcal{H}^p_\Gamma) \) boundedness of these operators, and then we obtain the \( L^p \) boundedness in the whole range \( 1 < p < \infty \) by interpolation again. The same holds for the other families of truncations \( \omega \). This finishes the proof of the Main Theorem 3.0.1.

Let us stress that almost all the estimates in the proof of Main Theorem 3.0.1 (in particular, the constants involved in the relationships \( \lesssim, \gtrsim \) and \( \approx \)) depend either on \( n, d, K \) or \( \text{Lip}(A) \), and possibly on other variables such as \( \rho \) or \( p \).

### 3.3 Proof of Theorem 3.2.1

In order to study the difference \( (K\tilde{\varphi}_{2^{-m}} \ast \mu)(x) - E_{m}\mu(x) \), we are going to split \( E_{m}\mu(x) \) into two parts, the one we will compare with \( (K\tilde{\varphi}_{2^{-m}} \ast \mu)(x) \) (which corresponds to integrating, in the definition of \( E_{m}\mu(x) \), over the points \( y \in \mathbb{R}^d \) such that \( 2^{-m} \lesssim \|\tilde{x} - \tilde{y}\| \), and the remaining part. Then, we will estimate each part of \( (K\tilde{\varphi}_{2^{-m}} \ast \mu)(x) - E_{m}\mu(x) \) separately, using the
cancellation properties of the kernel $K$ and the uniform rectifiability of $\mu$.

Recall from (3.1.18) that $E_m\mu(x) = \iint \Lambda^\mu_m(x,z,y)K(z-y)\,d\mu(z)\,d\mu(y)$. Given $\epsilon > 0$, we set $\gamma_\epsilon := 1 - \tilde{\varphi}_\epsilon$. Then,

$$E_m\mu(x) = \iint \tilde{\varphi}_{2^{-m}}(x-y)\Lambda^\mu_m(x,z,y)K(z-y)\,d\mu(z)\,d\mu(y) + \iint \gamma_{2^{-m}}(x-y)\Lambda^\mu_m(x,z,y)K(z-y)\,d\mu(z)\,d\mu(y).$$

The first term in the previous sum is the one that we will compare with $(K\tilde{\varphi}_{2^{-m}}*\mu)(x)$. For all $a \in \mathbb{R}^n$ such that $x \in D^a_m$, we have supp $\tilde{\varphi}_{2^{-m}}(x-\cdot) \cap D^a_m = \emptyset$, and thus $(K\tilde{\varphi}_{2^{-m}}*\mu)(x) = (K\tilde{\varphi}_{2^{-m}}*(\chi_{(D^a_m)\mu}))(x)$. Hence, using Fubini’s theorem and the definition of $\Lambda^\mu_m(x,z,y)$,

$$(K\tilde{\varphi}_{2^{-m}}*\mu)(x) = 2^m \int_{\{a:x \in D^a_m\}} (K\tilde{\varphi}_{2^{-m}}*(\chi_{(D^a_m)\mu}))(x)\,da = 2^m \int_{\{a:x \in D^a_m\}} \mu(D^a_m)^{-1} \int_{D^a_m} (K\tilde{\varphi}_{2^{-m}}*(\chi_{(D^a_m)\mu}))(x)\,d\mu(z)\,da = \iint \tilde{\varphi}_{2^{-m}}(x-y)\Lambda^\mu_m(x,z,y)K(x-y)\,d\mu(z)\,d\mu(y).$$

We can decompose $(K\tilde{\varphi}_{2^{-m}}*\mu)(x) - E_m\mu(x)$ as

$$(K\tilde{\varphi}_{2^{-m}}*\mu)(x) - E_m\mu(x) = \iint \tilde{\varphi}_{2^{-m}}(x-y)\Lambda^\mu_m(x,z,y)(K(x-y) - K(z-y))\,d\mu(z)\,d\mu(y) - \iint \gamma_{2^{-m}}(x-y)\Lambda^\mu_m(x,z,y)K(z-y)\,d\mu(z)\,d\mu(y) = \sum_{j<m} F^m_j(x) - \sum_{j \in \mathbb{Z}} G^m_j(x),$$

where

$$F^m_j(x) := \iint \tilde{\varphi}_{2^{-m-j}}^2(x-y)\Lambda^\mu_m(x,z,y)(K(x-y) - K(z-y))\,d\mu(z)\,d\mu(y), \quad (3.3.2)$$

$$G^m_j(x) := \iint \tilde{\varphi}_{2^{-m-j}}^2(x-y)\gamma_{2^{-m}}(x-y)\Lambda^\mu_m(x,z,y)K(z-y)\,d\mu(z)\,d\mu(y). \quad (3.3.3)$$

Fix a $v$-cube $D \in \mathcal{D}_m$, for some $m \in \mathbb{Z}$. In Subsection 3.3.1 (see (3.3.18)) we will prove that

$$\sum_{j<m} |F^m_j(x)| \lesssim \frac{\operatorname{dist}(x,L_D)}{\ell(D)} + \sum_{Q \in \mathcal{D} : \ell(D) \leq \ell(Q)} \frac{\ell(D)}{\ell(Q)} \alpha(Q), \quad (3.3.4)$$

for all $x \in D \cap \Gamma$, where $L_D$ denotes an $n$-plane that minimizes $\alpha(D)$, and in Subsection 3.3.2 (see (3.3.37)) we will prove that there exists a constant $C_b > 1$ such that

$$\sum_{j \in \mathbb{Z}} |G^m_j(x)| \lesssim C_b \alpha(D) + \sum_{Q \in \mathcal{D} : \ell(Q) \leq C_b \ell(D)} \frac{\ell(Q)^{n+1}}{\ell(D)^{n+1}} \alpha(Q). \quad (3.3.5)$$
for all $x \in D \cap \Gamma$. Assuming that these estimates hold, by (3.3.1),

$$\|W\mu\|_{L^2(\mu)}^2 = \sum_{m \in \mathbb{Z}} \sum_{D \in D_m} \int_D |(K\tilde{\varphi}_{2^{-m}} * \mu)(x) - E_m\mu(x)|^2 \, d\mu(x)$$

$$\lesssim \sum_{D \in \mathcal{D}} \int_D \left( \frac{\text{dist}(x, L_D)}{\ell(D)} \right)^2 d\mu(x) + \sum_{D \in \mathcal{D}} \left( \sum_{\beta \in D : \beta \subset \mathcal{Q}} \ell(D) \alpha(Q) \right)^2 \mu(D)$$

$$+ \sum_{D \in \mathcal{D}} \alpha(C_\beta D)^2 \mu(D) + \sum_{D \in \mathcal{D}} \left( \sum_{\beta \in D : \beta \subset \mathcal{Q}} \frac{\ell(Q)}{\ell(D)^{n+1}} \alpha(Q) \right)^2 \mu(D)$$

$$=: W_1\mu + W_2\mu + W_3\mu + W_4\mu.$$

If $L_D^1$ and $L_D^2$ denote a minimizing $n$-plane for $\beta_1(D)$ and $\beta_2(D)$, respectively, one can show that $\text{dist}_H(L_D \cap C_T, L_D^1 \cap C_T) \lesssim \alpha(D)\ell(D)$ and $\text{dist}_H(L_D^1 \cap C_T, L_D^2 \cap C_T) \lesssim \beta_2(D)\ell(D)$. This easily implies that, for $x \in D \cap \Gamma$, $\text{dist}(x, L_D) \lesssim \text{dist}(x, L_D^1) + \beta_2(D)\ell(D) + \alpha(D)\ell(D)$, so $W_1\mu \lesssim \sum_{D \in \mathcal{D}} (\alpha(D)^2 + \beta_2(D)^2)\mu(D)$.

By Cauchy-Schwarz inequality,

$$W_2\mu \leq \sum_{D \in \mathcal{D}} \mu(D) \left( \sum_{\beta \in D : \beta \subset \mathcal{Q}} \frac{\ell(D)}{\ell(Q)} \alpha(Q) \right)^2 \left( \sum_{\beta \in D : \beta \subset \mathcal{Q}} \frac{\ell(D)}{\ell(Q)} \right)$$

$$\approx \sum_{D \in \mathcal{D}} \sum_{\beta \in D : \beta \subset \mathcal{Q}} \frac{\ell(D)^{n+1}}{\ell(Q)} \alpha(Q)^2 \approx \sum_{\beta \in \mathcal{Q}} \alpha(Q)^2 \mu(Q),$$

and also

$$W_4\mu \leq \sum_{D \in \mathcal{D}} \mu(D) \left( \sum_{\beta \in D : \beta \subset \mathcal{Q}} \frac{\ell(Q)}{\ell(D)^{n+1}} \alpha(Q) \right)^2 \left( \sum_{\beta \in D : \beta \subset \mathcal{Q}} \frac{\ell(Q)}{\ell(D)^{n+1}} \right)$$

$$\approx \sum_{D \in \mathcal{D}} \sum_{\beta \in D : \beta \subset \mathcal{Q}} \frac{\ell(Q)^n \ell(Q)}{\ell(D)^2} \alpha(Q)^2 \lesssim \sum_{\beta \in \mathcal{Q}} \alpha(Q)^2 \mu(Q).$$

Therefore, using (3.3.6) and that $\alpha(Q) \lesssim C_\beta Q$, we conclude that

$$\|W\mu\|_{L^2(\mu)}^2 \lesssim \sum_{Q \in \mathcal{D}} (\alpha(C_\beta Q)^2 + \beta_2(Q)^2)\mu(Q),$$

and the theorem follows. It only remains to prove (3.3.4) and (3.3.5).

### 3.3.1 Estimate of $\sum_{j < m} F^m_j(x)$ when $x \in D \cap \Gamma$ for some $D \in \mathcal{D}_m$

Assume that $x \in D \cap \Gamma$ for some $D \in \mathcal{D}_m$ and $j < m$. Let $L_D$ be an $n$-plane that minimizes $\alpha(D)$ and let $\sigma_D := c_D \mathcal{H}^n_{L_D}$ be a minimizing measure of $\alpha(D)$. Let $L_D^j$ be the $n$-plane parallel to $L_D$ that contains $x$ and set $\sigma_D^j := c_D \mathcal{H}^n_{L_D^j}$. 

Notice that, because of \( x \in L_D \), the antisymmetry of \( \tilde{\varphi}_{2-j-1}^{2-j} K \), and since \( j < m \) (so, if \( x \in D_m^a \) and \( y \in \text{supp} \tilde{\varphi}_{2-j-1}^{2-j} (x - \cdot) \), then \( y \notin D_m^a \)), we have

\[
0 = \int \tilde{\varphi}_{2-j-1}^{2-j} (x-y) K (x-y) \, d\sigma_D^x (y) \\
= \int_{\{a: x \in D_m^a \}} \frac{2^{mn}}{\sigma_D^x (D_m^a)} \int_{D_m^a} \int_{(D_m^a)^c} \tilde{\varphi}_{2-j-1}^{2-j} (x-y) K (x-y) \, d\sigma_D^z (y) \, d\sigma_D^x (z) \, da \tag{3.3.7}
\]

\[
= \int \int \tilde{\varphi}_{2-j-1}^{2-j} (x-y) \Lambda_{m}^{a,x} (x, z ; y) K (x-y) \, d\sigma_D^z (z) \, d\sigma_D^x (y).
\]

Given \( a \in \mathbb{R}^n \), let \( b := a + \{2^{-m-1}\}^n \in \mathbb{R}^n \) be the center of \( \tilde{D}_m^a \). For \( u \in \mathbb{R}^n \) we denote \( \|u\|_{\infty} := \max_{i=1, \ldots, n} |u_i| \). Then, given \( t \in \mathbb{R}^d \), it is clear that \( t \in \overline{D_m^a} \) if and only if \( \|t-b\|_{\infty} \leq 2^{-m} \). Using that \( \sigma_D^x \) is a Hausdorff measure on an \( n \)-plane, that \( K \) is antisymmetric and that \( \tilde{\varphi}_{2-j-1}^{2-j} \) is symmetric, one can show that

\[
0 = \int_{\|x-b\|_{\infty} \leq 2^{-m}} \int_{\|x-b\|_{\infty} \leq 2^{-m}} \int_{\|y-b\|_{\infty} > 2^{-m}} \tilde{\varphi}_{2-j-1}^{2-j} (x-y) K(z-y) \, d\sigma_D^y (y) \, d\sigma_D^x (z) \, db.
\]

By the change of variable \( b = a + \{2^{-m-1}\}^n \), it is easy to see that this triple integral is equal to \( \int_{\{a: x \in D_m^a \}} \int_{D_m^a} \int_{(D_m^a)^c} \tilde{\varphi}_{2-j-1}^{2-j} (x-y) K (z-y) \, d\sigma_D^y (y) \, d\sigma_D^x (z) \, da \). Thus, since \( \sigma_D^x (D_m^a) \) does not depend on \( a \in \mathbb{R}^n \) because \( \sigma_D^x \) is flat,

\[
0 = \int_{\{a: x \in D_m^a \}} \frac{2^{mn}}{\sigma_D^x (D_m^a)} \int_{D_m^a} \int_{(D_m^a)^c} \tilde{\varphi}_{2-j-1}^{2-j} (x-y) K(z-y) \, d\sigma_D^y (y) \, d\sigma_D^x (z) \, da \tag{3.3.8}
\]

\[
= \int \int \tilde{\varphi}_{2-j-1}^{2-j} (x-y) \Lambda_{m}^{a,x} (x, z ; y) K (z-y) \, d\sigma_D^x (z) \, d\sigma_D^x (y).
\]

By (3.3.7) and (3.3.8), we conclude that

\[
0 = \int \int \tilde{\varphi}_{2-j-1}^{2-j} (x-y) \Lambda_{m}^{a,x} (x, z ; y) (K (x-y) - K (z-y)) \, d\sigma_D^x (z) \, d\sigma_D^x (y). \tag{3.3.9}
\]

By definition, it is clear that \( \Lambda_{m}^{a,x} (x, z ; y) = \Lambda_{m}^{a,y} (x, z ; y) \). Therefore, using (3.3.9), we can decompose

\[
F_j^m (x) = F_{1,j}^m (x) + F_{2,j}^m (x) + F_{3,j}^m (x) + F_{4,j}^m (x), \tag{3.3.10}
\]

where

\[
F_{1,j}^m (x) := \int \int \tilde{\varphi}_{2-j-1}^{2-j} (x-y) \Lambda_{m}^{a,x} (x, z ; y) (K (x-y) - K (z-y)) \, d(\mu - \sigma_D) (z) \, d\mu (y), \tag{3.3.11}
\]

\[
F_{2,j}^m (x) := \int \int \tilde{\varphi}_{2-j-1}^{2-j} (x-y) \Lambda_{m}^{a,x} (x, z ; y) (K (x-y) - K (z-y)) \, d\sigma_D (z) \, d(\mu - \sigma_D) (y), \tag{3.3.12}
\]
We have
\[ |\Lambda_m^\mu(x, z; y) - \Lambda_m^\sigma(x, z; y)| \leq 2^{j-m} \alpha(D), \]
(3.3.15)
\[ |F_2^m(x)| \lesssim 2^{j-m} \sum_{Q \in \mathcal{D} : D \subset Q, \ell(Q) \leq 2^{-j}} \alpha(Q), \]
(3.3.16)
\[ |F_3^m(x)| \lesssim 2^{j-m} \frac{\text{dist}(x, L_D)}{\ell(D)}. \]
(3.3.17)

Then, using (3.3.10), we will finally get that, for all \( D \in \mathcal{D}_m \) and \( x \in D \cap \Gamma \),
\[
\sum_{j<m} |F_j^m(x)| \lesssim \frac{\text{dist}(x, L_D)}{\ell(D)} + \sum_{j \leq m} 2^{j-m} \sum_{Q \in \mathcal{D} : D \subset Q, \ell(Q) \leq 2^{-j}} \alpha(Q)
\lesssim \frac{\text{dist}(x, L_D)}{\ell(D)} + \sum_{Q \in \mathcal{D} : D \subset Q} \frac{\ell(D)}{\ell(Q)} \alpha(Q),
\]
(3.3.18)
which gives (3.3.4).

### 3.3.1.1 Estimate of \( F_1^m(x) \)

Notice that, if \( |\bar{x} - \bar{z}| > 2^{-m} \sqrt{n} \), there is no \( a \in \mathbb{R}^n \) such that \( x, z \in D_m^a \), and this means that \( \Lambda_m^\mu(x, z; y) = 0 \). Thus, we can assume that \( |\bar{x} - \bar{z}| \leq 2^{-m} \sqrt{n} \). Therefore, if the constant \( C_\Gamma \) (see the definition of the \( \alpha \)'s in Subsection 3.1.2) is big enough, \( \text{supp} \Lambda_m^\mu(x, \cdot; y) \subset B_D \).

For \( y, z \in \Gamma \) such that \( y \in \text{supp} \bar{\varphi}_{2^{-j-1}}(x - \cdot) \), \( j < m \) and \( |\bar{x} - \bar{z}| \leq 2^{-m} \sqrt{n} \) (so, in particular, \( |x - z| \lesssim |x - y| \)), we have the following estimates:
\[
|K(x - y) - K(z - y)| \lesssim |x - z||x - y|^{-n-1} \lesssim 2^{j(n+1)-m},
\]
\[
|\nabla_z(K(x - y) - K(z - y))| = |\nabla_z K(z - y)| \lesssim 2^{j(n+1)}.
\]

**Claim 3.3.1.** We have \( |\Lambda_m^\mu(x, z; y)| \lesssim 2^{mn} \) and \( |\nabla_z \Lambda_m^\mu(x, z; y)| \lesssim 2^{m(n+1)} \) for all \( x, y, z \in \mathbb{R}^d \).
Claim 3.3.1 and the subsequent ones 3.3.2, . . . , 3.3.7 will be proved in Subsection 3.3.3 below.

Putting all these estimates together we obtain that
\[ |\nabla_z \left( \Lambda^\mu_m(x, z ; y)(K(x - y) - K(z - y)) \right) | \lesssim 2^{j(n+1)+mn}, \]
and, since \( \text{supp} \Lambda^\mu_m(x, \cdot ; y) \subset B_D \), recalling the definition of \( \text{dist}_{B_D} \) in (3.1.3),
\[ \left| \int \Lambda^\mu_m(x, z ; y)(K(x - y) - K(z - y)) d(\mu - \sigma_D)(z) \right| \lesssim 2^{j(n+1)+mn} \text{dist}_{B_D}(\mu, \sigma_D). \]

We can use this last estimate in (3.3.11) to obtain
\[ |F1_j^m(x)| \lesssim 2^{j(n+1)+mn} \text{dist}_{B_D}(\mu, \sigma_D) \int \varphi^{2-j}_{2-j,1}(x - y) d\mu(y) \]
\[ \lesssim 2^{j+m} \text{dist}_{B_D}(\mu, \sigma_D) \approx 2^{j-m} \ell(D)^{-n-1} \text{dist}_{B_D}(\mu, \sigma_D) \lesssim 2^{j-m} \alpha(D), \]
which, together with the estimate of \( |F3_j^m(x)| \) in Subsection 3.3.3.3, gives (3.3.15).

### 3.3.1.2 Estimate of \( F2_j^m(x) \)

Arguing as in Subsection 3.3.1.1, we can obtain the following estimates for \( x, y, z \) as above:
\[ |\tilde{\varphi}^{2-j}_{2-j,1}(x - y)| \leq 1 \quad \text{and} \quad |\nabla_y \tilde{\varphi}^{2-j}_{2-j,1}(x - y)| \lesssim 2^j, \quad (3.3.19) \]
\[ |K(x - y) - K(z - y)| \lesssim |x - z||x - y|^{-n-1} \lesssim 2^{j(n+1)-m}, \quad (3.3.20) \]
\[ |\nabla_y (K(x - y) - K(z - y))| \lesssim |x - z||x - y|^{-n-2} \lesssim 2^{j(n+2)-m}. \quad (3.3.21) \]
(by \( |\nabla_y K(x - y)| \lesssim 2^{j(n+2)} \) we mean that all the components of the matrix \( \nabla_y K(x - y) \) are bounded in absolute value by \( C2^{j(n+2)} \)).

**Claim 3.3.2.** For \( j < m, \ y \in \text{supp} \tilde{\varphi}^{2-j}_{2-j,1}(x - \cdot) \), and \( |\tilde{x} - \tilde{z}| \leq 2^{-m}\sqrt{n} \), the following hold:
\[ |\Lambda^\mu_m(x, z ; y)| \lesssim 2^{mn} \quad \text{and} \quad \nabla_y \Lambda^\mu_m(x, z ; y) = 0. \]

Notice that the first estimate in Claim 3.3.2 is the same as the first one in Claim 3.3.1.

Let \( D_j \in D_J \) be the unique dyadic v-cube with \( \ell(D_j) = 2^{-j} \) which contains \( D \). Then, \( \text{supp} \tilde{\varphi}^{2-j}_{2-j,1}(x - \cdot) \subset B_{D_j} \) for \( C \) big enough. Therefore, we can use the previous estimates to see that the gradient of the term inside the integral with respect to \( y \) in (3.3.12) is bounded by \( 2^{j(n+2)+m(n-1)} \) and is supported in \( B_{D_j} \), and then by (3.1.3) we derive that
\[ |F2_j^m(x)| \leq \int \left| \int \tilde{\varphi}^{2-j}_{2-j,1}(x - y)\Lambda^\mu_m(x, z ; y)(K(x - y) - K(z - y)) d(\mu - \sigma_D)(y) \right| d\sigma_D(z) \]
\[ \lesssim \int_{|\tilde{x} - \tilde{z}| \leq 2^{-m}\sqrt{n}} 2^{j(n+2)+m(n-1)} \text{dist}_{B_{D_j}}(\mu, \sigma_D) d\sigma_D(z) \]
\[ \lesssim 2^{j(n+2)-m} \text{dist}_{B_{D_j}}(\mu, \sigma_D). \quad (3.3.22) \]
We shall estimate \( \text{dist}_{B_D}(\mu, \sigma_D) \) in terms of the \( \alpha \) coefficients. Consider the unique sequence of dyadic \( v \)-cubes \( D =: D_m \subset \ldots \subset D_{i+1} \subset D_i \subset \ldots \subset D_j \) such that each \( D_i \) belongs to \( D_i \), for \( i = j, \ldots, m \). Let \( L_{D_i} \) be an \( n \)-plane that gives the minimum in the definition of \( \alpha(D_i) \) and let \( \sigma_{D_i} := c_{D_i} d\mathcal{H}_{L_D}^{n} \) be a minimizing measure. We will prove that

\[
\text{dist}_{B_D}(\mu, \sigma_D) \lesssim 2^{-j(n+1)} \sum_{i=j}^{m-1} \alpha(D_i).
\] (3.3.23)

Combining (3.3.23) with (3.3.22), we will finally obtain that \( |F_{2^m_j}(x)| \lesssim 2^{j-m} \sum_{i=j}^{m-1} \alpha(D_i) \), which gives (3.3.16).

Let us prove (3.3.23). By the triangle inequality,

\[
\text{dist}_{B_D}(\mu, \sigma_D) \leq \text{dist}_{B_D}(\mu, \sigma_{D_j}) + \sum_{i=j}^{m-1} \text{dist}_{B_D}(\sigma_{D_i}, \sigma_{D_{i+1}}) \lesssim 2^{-j(n+1)} \alpha(D_j) + \sum_{i=j}^{m-1} \text{dist}_{B_D}(\sigma_{D_i}, \sigma_{D_{i+1}}),
\]

so we are reduced to prove that, for all \( i = j, \ldots, m - 1 \),

\[
\text{dist}_{B_D}(\sigma_{D_i}, \sigma_{D_{i+1}}) \lesssim 2^{-j(n+1)} \alpha(D_i).
\] (3.3.24)

By definition, \( \text{dist}_{B_D}(\sigma_{D_i}, \sigma_{D_{i+1}}) = \sup \int g \, d(\mathcal{H}_{L_{D_i}}^{n} - c_{D_{i+1}} \mathcal{H}_{L_{D_{i+1}}}^{n}) \), where the supremum is taken over all Lipschitz functions \( g \) supported in \( B_D \) such that \( \text{Lip}(g) \leq 1 \). Fix one of such Lipschitz functions \( g \). Then,

\[
\int g \, d(\mathcal{H}_{L_{D_i}}^{n} - c_{D_{i+1}} \mathcal{H}_{L_{D_{i+1}}}^{n}) = (c_{D_i} - c_{D_{i+1}}) \int g \, d\mathcal{H}_{L_{D_i}}^{n} + c_{D_{i+1}} \int g \, d(\mathcal{H}_{L_{D_i}}^{n} - \mathcal{H}_{L_{D_{i+1}}}^{n}).
\] (3.3.25)

It is shown in [To11, Lemma 3.4] that \( |c_{D_i} - c_{D_{i+1}}| \lesssim \alpha(D_i) \), so the first term on the right hand side of (3.3.25) is bounded in absolute value by \( C2^{-j(n+1)} \alpha(D_i) \).

In order to estimate the second term of the right hand side of (3.3.25), set \( L_{D_{i+1}} = \{ (\tilde{t}, a(\tilde{t})) : \tilde{t} \in \mathbb{R}^d \} \) (where \( a : \mathbb{R}^n \to \mathbb{R}^{d-n} \) is an appropriate affine map), and let \( p : L_{D_i} \to L_{D_{i+1}} \) be the projection defined by \( p(t) := (\tilde{t}, a(\tilde{t})) \). Since \( \Gamma \) is a Lipschitz graph, \( a \) is well defined and \( p \) is a homeomorphism. Let \( p_t \mathcal{H}_{L_{D_i}}^{n} \) be the image measure of \( \mathcal{H}_{L_{D_i}}^{n} \) by \( p \). It is easy to check that \( \mathcal{H}_{L_{D_{i+1}}}^{n} = \tau p_t \mathcal{H}_{L_{D_i}}^{n} \), where \( \tau \) is some positive constant such that \( |\tau - 1| \lesssim \alpha(D_i) \) and \( \tau \lesssim 1 \). Therefore,

\[
\left| \int g \, d(\mathcal{H}_{L_{D_i}}^{n} - \mathcal{H}_{L_{D_{i+1}}}^{n}) \right| = \left| \int (g(t) - \tau g(p(t))) \, d\mathcal{H}_{L_{D_i}}^{n}(t) \right|
\leq \left| \int (1 - \tau)g(t) \, d\mathcal{H}_{L_{D_i}}^{n}(t) \right| + \left| \int \tau g(t) - g(p(t)) \, d\mathcal{H}_{L_{D_i}}^{n}(t) \right|
\lesssim 2^{-j(n+1)} \alpha(D_i) + \int |(g(t) - g(p(t))) \, d\mathcal{H}_{L_{D_i}}^{n}(t)|.
\] (3.3.26)
To prove the theorem, let \( g \) and \( g \circ p \) be supported in \( B_D \) and \( g \) be 1-Lipschitz, by [To11, Lemma 3.4],

\[
\int |(g - g \circ p)| \, d\mathcal{H}^n_{L_D} \lesssim \int_{B_D} \text{dist}_{\mathcal{H}}(L_D, L_D \cap B_D) \, d\mathcal{H}^n_{L_D} \lesssim 2^{- j n} \text{dist}_{\mathcal{H}}(L_D, L_D \cap B_D) \lesssim 2^{- j(n+1)} \alpha(D).
\]

This last estimate together with (3.3.26) and the fact that \(|c_{D+i}| \lesssim 1\) implies that the second term on the right hand side of (3.3.25) is also bounded in absolute value by \( C 2^{- j(n+1)} \alpha(D)\). Therefore, to obtain (3.3.24) we only have to take the supremum in (3.3.25) over all admissible functions \( g \).

### 3.3.1.3 Estimate of \( F^{3m}_j(x) \)

Notice that, by Fubini’s theorem,

\[
\Lambda^a_m(x, z \mid y) - \Lambda^\sigma_m(x, z \mid y) = 2^{mn} \int_{\{a : x, z \in D_m^a, y \notin D_m^a\}} \left( \frac{1}{\mu(D_m^a)} - \frac{1}{\sigma(D_m^a)} \right) \, da
\]

\[
= 2^{mn} \int_{\{a : x, z \in D_m^a, y \notin D_m^a\}} \frac{\sigma(D_m^a) - \mu(D_m^a)}{\mu(D_m^a) \sigma(D_m^a)} \, da
\]

\[
= 2^{mn} \int_{\{a : x, z \in D_m^a, y \notin D_m^a\}} \left( \int_{t \in D_m^a} d(\sigma_D - \mu)(t) \right) \mu(D_m^a)^{-1} \sigma(D_m^a)^{-1} \, da
\]

\[
= \int \Lambda^\mu\sigma_m(x, z, t \mid y) \, d(\sigma_D - \mu)(t).
\]

Since \( \Lambda^a_m(x, z, t \mid y) = 0 \) if \(|x - \tilde{t}| > 2^{- m} \sqrt{n}\), we may assume that \( \text{supp} \Lambda^\mu\sigma_m(x, z, t \mid y) \subset B_D \) (by taking \( C_\Omega \) big enough).

**Claim 3.3.3.** We have \(|\Lambda^\mu\sigma_m(x, z, t \mid y)| \lesssim 2^{2mn}\) and \(|\nabla_t \Lambda^\mu\sigma_m(x, z, t \mid y)| \lesssim 2^{m(2n+1)}\) for all \( x, y, z, t \in \mathbb{R}^d \).

Using Claim 3.3.3, we deduce that \(|\Lambda^a_m(x, z \mid y) - \Lambda^\sigma_m(x, z \mid y)| \lesssim 2^{m(2n+1)} \text{dist}_{B_D}(\mu, \sigma_D)\), and then,

\[
|F^{3m}_j(x)| \lesssim \int \bar{\varphi}_{2^{- j-1}}(x - y) \int_{|z| \leq 2^{- m} \sqrt{n}} 2^{m(2n+1)} \text{dist}_{B_D}(\mu, \sigma_D) \, |x - z| |x - y|^{- n-1} \, d\sigma_D(z) \, d\sigma_D(y)
\]

\[
\lesssim 2^{mn+j(n+1)} \text{dist}_{B_D}(\mu, \sigma_D) \int \int_{|z| \leq 2^{- j3\sqrt{n}}} |d\sigma_D(z)| \, d\sigma_D(y)
\]

\[
\lesssim 2^{mn+j} \text{dist}_{B_D}(\mu, \sigma_D) \lesssim 2^{- m} \alpha(D),
\]

which, together with the estimate of \(|F^{1m}_j(x)|\) (see Subsection 3.3.1.1), gives (3.3.15).
3.3.1.4 Estimate of $F_4^m(x)$

Set $L_D = \{(\overline{y}, a(\overline{y})) \in \mathbb{R}^d : \overline{y} \in \mathbb{R}^n \}$, where $a : \mathbb{R}^n \to \mathbb{R}^{d-n}$ is an appropriate affine map, and let $p : L_D^c \to L_D$ be the projection defined by $p(y) := (\overline{y}, a(\overline{y}))$. Since $\Gamma$ is a Lipschitz graph, $a$ is well defined and $p$ is a homeomorphism. If $p_\ast \sigma_D^x$ is the image measure of $\sigma_D^x$ under $p$, we obviously have $\sigma_D = p_\ast \sigma_D^x$ because $L_D$ and $L_D^c$ differ from a translation. Therefore, since $\overline{p}(y) = \overline{y}$, (3.3.14) becomes

$$F_4^m(x) = \int \left( K(x - p(y)) - K(p(z) - p(y)) - (K(x - y) - K(z - y)) \right).$$

For $y, z \in L_D^c$ such that $\overline{\varphi}_{2^{-j-1}}(x - y) \Lambda_m^\sigma(x, z ; y) \neq 0$, we have $K(p(z) - p(y)) - K(z - y) = 0$, so we can estimate

$$|K(x - p(y)) - K(p(z) - p(y)) - (K(x - y) - K(z - y))| = |K(x - p(y)) - K(x - y)| \lesssim \frac{|y - p(y)|}{|x - y|^{n+1}} \lesssim 2^{j(n+1)}|y - p(y)| \approx 2^{j(n+1)} \text{dist}(x, L_D).$$

By the same arguments as in the proof of Claim 3.3.1, one can see that $|\Lambda_m^\sigma(x, z ; y)| \lesssim 2^m$. Therefore,

$$|F_4^m(x)| \lesssim 2^{j(n+1)} \text{dist}(x, L_D) 2^{mn} \int \int_{|\overline{x} - \overline{y}| \leq 2^{-j-3} \sqrt{n}} d\sigma_D^x(z) d\sigma_D^x(y) \lesssim 2^j \text{dist}(x, L_D) \approx 2^{j-m} \text{dist}(x, L_D)/\ell(D),$$

which gives (3.3.17).

3.3.2 Estimate of $\sum_{j \in \mathbb{Z}} G_j^m(x)$ when $x \in D \cap \Gamma$ for some $D \in \mathcal{D}_m$

Assume that $x \in D$ for some $D \in \mathcal{D}_m$. Recall from (3.3.3) that

$$G_j^m(x) = \int \int \tilde{\varphi}_{2^{-j-1}}(z - y) \gamma_{2^{-m}}(x - y) \Lambda_m^\mu(x, z ; y) K(z - y) d\mu(z) d\mu(y),$$

where $0 \leq \gamma_{2^{-m}}(x - y) \leq 1$, $|\nabla_y \gamma_{2^{-m}}(x - y)| \lesssim 2^m$ for all $x, y \in \mathbb{R}^d$, and $\gamma_{2^{-m}}(x - y) = 0$ whenever $|\overline{x} - \overline{y}| > 2^{-m} \sqrt{n}$. Notice that $\Lambda_m^\mu(x, z ; y) = 0$ if $|\overline{x} - \overline{z}| > 2^{-m} \sqrt{n}$, thus we can assume that $|\overline{x} - \overline{z}| \leq 2^{-m} \sqrt{n}$ and $|\overline{z} - \overline{y}| \leq 2^{-m+2} \sqrt{n}$ in the integral that defines $G_j^m(x)$. Hence, if $j \leq m - 2$, $\tilde{\varphi}_{2^{-j-1}}(z - y) \Lambda_m^\mu(x, z ; y) = 0$ for all $z, y \in \mathbb{R}^d$, because $\tilde{\varphi}_{2^{-j-1}}(z - y) = 0$ if $|\overline{z} - \overline{y}| \leq 2^{-j-1} \sqrt{n}$ and $2^{-j-1} \sqrt{n} \geq 2^{-m+2} \sqrt{n}$ when $j \leq m - 2$. Therefore, $G_j^m(x) = 0$ for $j \leq m - 2$, and then

$$\sum_{j \in \mathbb{Z}} G_j^m(x) = \sum_{j \geq m-1} G_j^m(x);$$

(3.3.27)
so, from now on, we assume that \( j \geq m - 1 \).

Let \( L_D \) be an \( n \)-plane that minimizes \( \alpha(D) \) and let \( \sigma_D := c_D \mathcal{H}_D^n \) be a minimizing measure of \( \alpha(D) \). As we did in the beginning of Subsection 3.3.1, given \( a \in \mathbb{R}^n \), let \( b := a + \{2^{-m-1}\}^n \in \mathbb{R}^n \) be the center of \( \tilde{D}_m^a \). Recall that, for \( t \in \mathbb{R}^d \), \( t \in \tilde{D}_m^a \) if and only if \( \|t - b\|_\infty \leq 2^{-m} \). Using that \( \sigma_D \) is a Hausdorff measure on an \( n \)-plane, that \( K \) is antisymmetric and that \( \tilde{\varphi}_{2^{-1},1} \) and \( \gamma_{2^{-m}} \) are symmetric, one can show that

\[
0 = \int_{\|\bar{e} - b\|_\infty \leq 2^{-m}} \int_{\|\bar{e} - b\|_\infty \leq 2^{-m}} \int_{\|\bar{y} - b\|_\infty > 2^{-m}} \tilde{\varphi}_{2^{-j}, 1}(z - y) \gamma_{2^{-m}}(x - y) K(z - y) d\sigma_D(y) d\sigma_D(z) db.
\]

By the change of variable \( b = a + \{2^{-m-1}\}^n \), it is easy to see that this triple integral is equal to

\[
\int_{\{a : x \in \tilde{D}_m^a\}} \int_{\tilde{D}_m^a} \int_{\tilde{D}_m^a} \tilde{\varphi}_{2^{-j}, 1}(z - y) \gamma_{2^{-m}}(x - y) K(z - y) d\sigma_D(y) d\sigma_D(z) da,
\]

and thus, since \( \sigma_D(D_m^a) \) does not depend on \( a \in \mathbb{R}^n \) because \( \sigma_D \) is flat,

\[
0 = \int_{\{a : x \in \tilde{D}_m^a\}} \frac{2^m}{\sigma_D(D_m^a)} \int_{\tilde{D}_m^a} \int_{\tilde{D}_m^a} \tilde{\varphi}_{2^{-j}, 1}(z - y) \gamma_{2^{-m}}(x - y) K(z - y) d\sigma_D(y) d\sigma_D(z) da
\]

\[
= \int \tilde{\varphi}_{2^{-j}, 1}(z - y) \gamma_{2^{-m}}(x - y) A_{\sigma_D}(x, z ; y) K(z - y) d\sigma_D(z) d\sigma_D(y).
\]

(3.3.28)

Let \( \{\eta_Q\}_{Q \in \mathcal{D}_j} \) be a partition of the unity with respect to the v-cubes \( Q \in \mathcal{D}_j \), i.e. \( \eta_Q : \mathbb{R}^d \to \mathbb{R} \) are \( C^\infty \) functions such that: \( \chi_{0.9Q} \leq \eta_Q \leq \chi_{1.1Q}, |\nabla \eta_Q| \lesssim \ell(Q)^{-1} = 2^j, \sum_{Q \in \mathcal{D}_j} \eta_Q = 1 \) and \( \eta_Q(y) = \eta_Q(\gamma_{2^{-j}}(y)) \) for all \( y \in \mathbb{R}^d \). It is easy to check that, if \( j \geq m - 1 \), \( Q \in \mathcal{D}_j \), and \( \text{supp} \eta_Q \cap \text{supp} \gamma_{2^{-m}}(x - \cdot) \neq \emptyset \), then \( Q \subset C \in \tilde{D}_e \) for some absolute constant \( C_\epsilon > 1 \).

Given \( Q \in \mathcal{D}_j \), let \( L_Q \) and \( \sigma_Q := c_Q \mathcal{H}_L^n \) be a minimizing \( n \)-plane and measure for \( \alpha(Q) \), respectively, and consider the measure

\[
\lambda := \sum_{Q \in \mathcal{D}_j \cap \mathcal{C}_e \subset D} \eta_Q \sigma_Q.
\]

By (3.3.28) and the properties of the partition of the unity \( \{\eta_Q\}_{Q \in \mathcal{D}_j} \), for \( j \geq m - 1 \) we can decompose \( G_j^m(x) \) as

\[
G_j^m(x) = G1_j^m(x) + G2_j^m(x) + G3_j^m(x) + G4_j^m(x) + G5_j^m(x),
\]

(3.3.29)

where

\[
G1_j^m(x) := \sum_{Q \in \mathcal{D}_j \cap \mathcal{C}_e \subset D} \int \ldots \mu d(\mu - \sigma_Q)(z) d\mu(y),
\]

(3.3.30)

\[
G2_j^m(x) := \sum_{Q \in \mathcal{D}_j \cap \mathcal{C}_e \subset D} \int \ldots \sigma_Q d(\mu - \sigma_Q)(y),
\]

(3.3.31)

\[
G3_j^m(x) := \sum_{Q \in \mathcal{D}_j \cap \mathcal{C}_e \subset D} \int \ldots \sigma_Q \times \sigma_Q d(\sigma - \sigma_D)(z)(y),
\]

(3.3.32)
where \( \ldots \) stands for \( \tilde{\varphi}_{2^{-j-1}}^2(z - y)\gamma_{2^{-m}}(x-y)\eta_Q(y)K(z-y)\Lambda_{m}^\mu(x, z; y)\), and

\[
G_{1j}^m(x) := \int \int \tilde{\varphi}_{2^{-j-1}}^2(z-y)\gamma_{2^{-m}}(x-y)K(z-y)(\Lambda_{m}^\mu(x, z; y) - \Lambda_{m}^\lambda(x, z; y)) \, d\sigma_D(z) \, d\sigma_D(y),
\]

(3.3.33)

\[
G_{5j}^m(x) := \int \int \tilde{\varphi}_{2^{-j-1}}^2(z-y)\gamma_{2^{-m}}(x-y)K(z-y)(\Lambda_{m}^\lambda(x, z; y) - \Lambda_{m}^{\sigma_D}(x, z; y)) \, d\sigma_D(z) \, d\sigma_D(y).
\]

(3.3.34)

In the next subsections we will prove the following estimates:
Therefore, for \( y \in \text{supp}(\eta_Q) \),
\[
\left| \int \tilde{\varphi}_{2^{-j},-1}^2(z - y)K(z - y)\Lambda^\mu_m(x, z; y) \, d(\mu - \sigma_Q)(z) \right| \lesssim 2^{m(n+1)+jn} \text{dist}_{B_\alpha}(\mu, \sigma_Q) \\
\lesssim 2^{m(n+1) - j} \alpha(Q),
\]
and then,
\[
|G_1^m(x)| \lesssim \sum_{Q \in D_j : Q \subset C_j D} \int_{\text{supp}(\eta_Q)} 2^{m(n+1) - j} \alpha(Q) \, d\mu(y) \lesssim \sum_{Q \in D_j : Q \subset C_j D} 2^{(m-j)(n+1)} \alpha(Q).
\]

### 3.3.2.2 Estimate of \( G_2^m_j(x) \)

It can be estimated using the arguments of Subsection 3.3.2.1, but now we also have to take into account that \( |\nabla y \gamma_{2^{-m}}(x - y)| \lesssim 2^m \lesssim 2^j \), because we are assuming \( j \geq m - 1 \), and we have to use the last estimate in Claim 3.3.4.

#### 3.3.2.3 Estimate of \( G_3^m_j(x) \)

Given \( x \in D \cap \Gamma \) and \( Q \in D_j \), denote
\[
H_Q(y, z) := \tilde{\varphi}_{2^{-j},-1}^2(z - y)\gamma_{2^{-m}}(x - y)\eta_Q(y)K(z - y)\Lambda^\mu_m(x, z; y).
\]

Then, (3.3.32) becomes
\[
G_3^m_j(x) = \sum_{Q \in D_j : Q \subset C_j D} \int H_Q(y, z) \, d(\sigma_Q \times \sigma_{-D} \times \sigma_{+D})(z, y) \\
= \sum_{Q \in D_j : Q \subset C_j D} \int \int H_Q(y, z) \, d(c_Q^2 H_{L_Q} \times H_{L_Q} - c_D^2 H_{L_Q} \times H_{L_Q}) \, d\mathcal{H}_{L_Q}^n(z) \, d\mathcal{H}_{L_Q}^n(y) \\
= \sum_{Q \in D_j : Q \subset C_j D} \int \int H_Q(y, z) \, d(H_{L_Q}^n \times H_{L_Q}^n - H_{L_Q}^n \times H_{L_Q}^n) \, d\mathcal{H}_{L_Q}^n(z, y) \tag{3.3.38}
\]
\[
=: G_3A^m_j(x) + G_3B^m_j(x).
\]

We are going to estimate the terms \( G_3A^m_j(x) \) and \( G_3B^m_j(x) \) separately. Recall that \( \ell(D) = 2^{-m} \). Given a v-cube \( Q \in D_j \) such that \( Q \subset C_j D \), let \( Q =: Q_j \subset \ldots \subset Q_{i+1} \subset Q_i \subset \ldots \subset Q_{m-1} \) be the sequence of v-cubes such that \( Q_i \) belongs to \( D_i \) for \( i = m - 1, \ldots, j \). Evidently, \( Q_{m-1} \subset C_b D \) for some constant \( C_b \) big enough, because \( \ell(Q_{m-1}) = 2 \ell(D) \) and \( Q \subset Q_{m-1} \cap C_e D \). Let \( L_{Q_i} \) be an \( n \)-plane that minimizes \( \alpha(Q_i) \) and let \( \sigma_{Q_i} := c_Q \mathcal{H}_{L_Q}^n \), be a minimizing measure of \( \alpha(Q_i) \). Also, let \( L_{C_b D} \) and \( \sigma_{C_b D} := c_{C_b D} \mathcal{H}_{L_{C_b D}}^n \) be a minimizing \( n \)-plane and measure of \( \alpha(C_b D) \), respectively.
In order to estimate $G3A_j^m(x)$, notice that, by [To11, Lemma 3.4] and the triangle inequality, $|c_{Q_i}| \leq 1$ for all $i = m - 1, \ldots, j$, and

$$|c_Q^2 - c_D^2| = |c_Q + c_D||c_Q - c_D| \lesssim |c_{Q_j} - c_D|$$

$$\lesssim |c_{Q_{m-1}} - c_{C_bD}| + |c_{C_bD} - c_D| + \sum_{i=m-1}^{j-1} |c_{Q_{i+1}} - c_{Q_i}|$$

$$\lesssim \alpha(C_bD) + \sum_{i=m-1}^{j-1} \alpha(Q_i) \lesssim \alpha(C_bD) + \sum_{R \in \mathcal{D} : Q \subset R \subset C_bD} \alpha(R)$$

(3.3.39)

(in the case that $j = m - 1$, there are no intermediate scales between $j$ and $m - 1$, so one just obtains $|c_Q^2 - c_D^2| \lesssim \alpha(C_bD)$).

**Claim 3.3.5.** For $z \in \text{supp} \tilde{\Psi}_{2^{-j-1}}^\sim (\cdot - y)$, we have $|\Lambda^y_m(x, z ; y)| \lesssim 2^{m(n+1)-j}.$

Notice that this last estimate is the same as the first one in Claim 3.3.4. Using Claim 3.3.5 and that $|K(z - y)| \lesssim 2^m$ for all $z \in \text{supp} \tilde{\Psi}_{2^{-j-1}}^\sim (\cdot - y)$, we easily obtain $|H_Q(y, z)| \lesssim 2^{m(n+1)+j(n-1)}$. Therefore, using (3.3.39),

$$|G3A_j^m(x)| \lesssim \sum_{Q \in \mathcal{D}_j : Q \subset C_bD} |c_Q^2 - c_D^2| \iint |H_Q(y, z)| d\mathcal{H}^n_{L_Q}(z) d\mathcal{H}^n_{L_Q}(y)$$

$$\lesssim \sum_{Q \in \mathcal{D}_j : Q \subset C_bD} 2^{(m-j)(n+1)} \left( \alpha(C_bD) + \sum_{R \in \mathcal{D} : Q \subset R \subset C_bD} \alpha(R) \right).$$

(3.3.40)

To estimate $G3B_j^m(x)$ in (3.3.38), we set

$$G3B_j^m(x) = \sum_{Q \in \mathcal{D}_j : Q \subset C_bD} c_D^2 \mathcal{G}3B(Q)_j^m(x),$$

(3.3.41)

where $G3B(Q)_j^m(x) := \iint H_Q d(\mathcal{H}^n_{L_Q} \times \mathcal{H}^n_{L_Q} - \mathcal{H}^n_{L_Q} \times \mathcal{H}^n_{L_Q})$. Given $Q \in \mathcal{D}_j$ such that $Q \subset C_bD$, we split $G3B(Q)_j^m(x)$ as follows:

$$G3B(Q)_j^m(x) = \sum_{i=m-1}^{j-1} \iint H_Q d(\mathcal{H}^n_{L_{Q_i+1}} \times \mathcal{H}^n_{L_{Q_i+1}} - \mathcal{H}^n_{L_{Q_i}} \times \mathcal{H}^n_{L_{Q_i}})$$

$$+ \iint H_Q d(\mathcal{H}^n_{L_{Q_m-1}} \times \mathcal{H}^n_{L_{Q_m-1}} - \mathcal{H}^n_{L_{C_bD}} \times \mathcal{H}^n_{L_{C_bD}})$$

$$+ \iint H_Q d(\mathcal{H}^n_{L_{C_bD}} \times \mathcal{H}^n_{L_{C_bD}} - \mathcal{H}^n_{L_D} \times \mathcal{H}^n_{L_D}).$$

(3.3.42)

(as before, in the case $j = m - 1$ the first term on the right hand side of (3.3.42) does not exist).

Fix $i \in \mathbb{Z}$ such that $m - 1 \leq i < j$. Set $L_{Q_{i+1}} = \{ (\tilde{y}, a(\tilde{y})) : \tilde{y} \in \mathbb{R}^n \}$, where $a : \mathbb{R}^n \to \mathbb{R}^{d-n}$ is an appropriate affine map, and let $p : L_{Q_i} \to L_{Q_{i+1}}$ be the map defined by
3.3. Proof of Theorem 3.2.1 \[ 65 \]

\[ p(y) := (\tilde{y}, a(\tilde{y})). \] Let \( p_2^q \mathcal{H}^a_{L_Q} \) be the image measure of \( \mathcal{H}^a_{L_Q} \) by \( p \). It is easy to check that \( \mathcal{H}^n_{L_{Q+1}} = p_2^q \mathcal{H}^a_{L_Q} \), where \( \tau \) is some positive constant such that \( |\tau - 1| \lesssim \alpha(Q_i) \) and \( \tau \lesssim 1 \). Therefore,

\[
\int \circ \int H_Q(y, z) d(\mathcal{H}^n_{L_{Q+1}} \times \mathcal{H}^n_{L_{Q+1}} - \mathcal{H}^n_{L_Q} \times \mathcal{H}^n_{L_Q})(z, y)
\]

\[
= \int \circ \int \tau^2 H_Q(p(y), p(z)) - H_Q(y, z) d\mathcal{H}^n_{L_Q}(z) d\mathcal{H}^n_{L_Q}(y)
\]

\[
= \int \circ \int \tau^2 \left( H_Q(p(y), p(z)) - H_Q(y, z) \right) d\mathcal{H}^n_{L_Q}(z) d\mathcal{H}^n_{L_Q}(y)
\]

\[
+ \int \circ \int (\tau^2 - 1) H_Q(y, z) d\mathcal{H}^n_{L_Q}(z) d\mathcal{H}^n_{L_Q}(y).
\]

(3.3.43)

Since \( |\tau^2 - 1| \lesssim \alpha(Q_i) \) and we have seen that \( |H_Q(y, z)| \lesssim 2^{(n+1)+j(n-1)} \) after Claim 3.3.5, the second term on the right side of the last equality is bounded by \( C2^{(m-j)(n+1)}\alpha(Q_i) \).

In order to estimate the first term on the right hand side of (3.3.43), notice that \( \tilde{\mathcal{H}}^n_{2-j} (z - y), \gamma_{2-m}(x - y), \eta_Q(y) \) and \( \Lambda_m(x, z ; y) \) only depend on the first \( n \) coordinates of \( y \) and \( z \) (i.e., on \( \tilde{y} \) and \( \tilde{z} \)), thus their values coincide on \( (y, z) \) and \( (p(y), p(z)) \). Then,

\[
\int \circ \int \tau^2 \left( H_Q(p(y), p(z)) - H_Q(y, z) \right) d\mathcal{H}^n_{L_Q}(z) d\mathcal{H}^n_{L_Q}(y)
\]

\[
= \tau^2 \int \circ \int \tilde{\mathcal{H}}^n_{2-j} (z - y) \gamma_{2-m}(x - y) \eta_Q(y) \Lambda_m(x, z ; y)
\]

\[
\left( K(p(z) - p(y)) - K(z - y) \right) d\mathcal{H}^n_{L_Q}(z) d\mathcal{H}^n_{L_Q}(y).
\]

Let \( \theta_i \) be the angle between \( L_Q \) and \( L_{Q+1} \). One can easily see that, for \( y, z \in L_Q \), \(|p(z) - p(y)| - (z - y)| \lesssim \sin(\theta_i)|z - y| \lesssim \alpha(Q_i)|z - y|. \) Thus, if also \( z \in \text{supp} \tilde{\mathcal{H}}^n_{2-j}(-y) \),

\[
|K(p(z) - p(y)) - K(z - y)| \lesssim 2^{j(n+1)}(p(z) - p(y)) - (z - y)| \lesssim 2^{jn} \alpha(Q_i).
\]

Together with Claim 3.3.5 and the fact that \( \tau^2 \lesssim 1 \), this gives

\[
\left| \int \circ \int \tau^2 \left( H_Q(p(y), p(z)) - H_Q(y, z) \right) d\mathcal{H}^n_{L_Q}(z) d\mathcal{H}^n_{L_Q}(y) \right|
\]

\[
\lesssim \int \circ \int \tilde{\mathcal{H}}^n_{2-j} (z - y) \eta_Q(y) 2^{m(n+1)+j(n-1)} \alpha(Q_i) d\mathcal{H}^n_{L_Q}(z) d\mathcal{H}^n_{L_Q}(y) \lesssim 2^{(m-j)(n+1)} \alpha(Q_i).
\]

From the last estimates and (3.3.43), we get

\[
\left| \int \circ \int H_Q \left( \mathcal{H}^n_{L_{Q+1}} \times \mathcal{H}^n_{L_{Q+1}} - \mathcal{H}^n_{L_Q} \times \mathcal{H}^n_{L_Q} \right) \right| \lesssim 2^{(m-j)(n+1)} \alpha(Q_i)
\]

for \( i = m - 1, \ldots, j - 1 \). With similar arguments, one also obtains

\[
\left| \int \circ \int H_Q \left( \mathcal{H}^n_{L_{Q+1}} \times \mathcal{H}^n_{L_{Q+1}} - \mathcal{H}^n_{L_{Q+1}} \times \mathcal{H}^n_{L_{Q+1}} \right) \right| \lesssim 2^{(m-j)(n+1)} \alpha(C_bD),
\]

\[
\left| \int \circ \int H_Q \left( \mathcal{H}^n_{L_{Q+1}} \times \mathcal{H}^n_{L_{Q+1}} - \mathcal{H}^n_{L_{Q+1}} \times \mathcal{H}^n_{L_{Q+1}} \right) \right| \lesssim 2^{(m-j)(n+1)} \alpha(C_bD).
\]
These last three inequalities together with (3.3.42), (3.3.41) and the fact that $|c_D| \lesssim 1$ yield

\[
|G3B_j^m(x)| \lesssim \sum_{Q \in D_j : Q \subset C_jD} 2^{(m-j)(n+1)} \left( \alpha(C_bD) + \sum_{i=m-1}^{j-1} \alpha(Q_i) \right) \\ \leq \sum_{Q \in D_j : Q \subset C_jD} \sum_{R \in D : Q \subset R \subset C_jD} 2^{(m-j)(n+1)} \left( \alpha(C_bD) + \sum_{R \in D : Q \subset R \subset C_jD} \alpha(R) \right).
\]

(3.3.44)

Finally, (3.3.40) and (3.3.44) applied to (3.3.38) give half of (3.3.36).

3.3.2.4 Estimate of $G4_j^m(x)$

By Fubini’s theorem and the definitions of $\lambda$, $\Lambda_m^\alpha$ and $\Lambda_m^\lambda$,

\[
\Lambda_m^\alpha(x, z; y) - \Lambda_m^\lambda(x, z; y) = 2^{mn} \int_{\{a : x \in D_m^n, y \notin D_m^n\}} \frac{\lambda(D_m^n) - \mu(D_m^n)}{\mu(D_m^n)\lambda(D_m^n)} \eta(t) d(\sigma_Q - \mu)(t) \]

\[
= 2^{mn} \sum_{Q \in D_j : Q \subset C_jD} \int_{\{a : x \in D_m^n, y \notin D_m^n\}} \eta(t) \Lambda_m^{\mu, \lambda}(x, z, t; y) d(\sigma_Q - \mu)(t).
\]

We also used in the second equality that $1 = \sum_{Q \in D_j} \eta_Q(t) = \sum_{Q \in D_j : Q \subset C_jD} \eta_Q(t)$ for all $t \in D_m^n$ if $C_e$ is big enough, and this is because $j \geq m - 1$ and $|\bar{x} - \bar{t}| \lesssim 2^{-m}$ for all $t \in D_m^n$.

Claim 3.3.6. For $x \in D$, $j \geq m - 1$, $|x - y| \lesssim 2^{-m}$, and $z \in \text{supp} \tilde{\varphi}_{2^{-j-1}}(-y)$, the following hold: $|\Lambda_m^{\mu, \lambda}(x, z, t; y)| \lesssim 2^{m(2n+1)-j}$ and $|\nabla_t \Lambda_m^{\mu, \lambda}(x, z, t; y)| \lesssim 2^{m(2n+1)}$.

Using Claim 3.3.6 and the properties of $\eta_Q$, we obtain

\[
|\Lambda_m^\alpha(x, z; y) - \Lambda_m^\lambda(x, z; y)| \lesssim \sum_{Q \in D_j : Q \subset C_jD} 2^{m(2n+1)} \text{dist}_D(\mu, \sigma_Q) \lesssim \sum_{Q \in D_j : Q \subset C_jD} 2^{m(2n+1)-j(n+1)} \alpha(Q).
\]

Plugging this estimate into the definition of $G4_j^m(x)$ in (3.3.33), we get

\[
|G4_j^m(x)| \lesssim \int \tilde{\varphi}_{2^{-j-1}}(z - y) |K(z - y)| \sum_{Q \in D_j : Q \subset C_jD} 2^{m(2n+1)-j(n+1)} \alpha(Q) d\sigma_D(z) d\sigma_D(y)
\]

\[
\lesssim \sum_{Q \in D_j : Q \subset C_jD} 2^{(m-j)(n+1)} \alpha(Q),
\]

which, together with the estimates of $|G1_j^m(x)|$ and $|G2_j^m(x)|$ in Subsections 3.3.2.1 and 3.3.2.2, gives (3.3.35).
3.3.2.5 Estimate of $G_2^m_j(x)$

Arguing as in Subsection 3.3.2.4, we have

$$
\Lambda_m^\lambda(x, z ; y) - \Lambda_m^{\sigma_D}(x, z ; y) = \sum_{Q \in D_j; Q \subset C_b D} \int \eta_Q(t) \Lambda_m^{\lambda, \sigma_D}(x, z, t ; y) d(\sigma_D - \sigma_Q)(t)
$$

(3.3.45)

where we have set $H_Q(t) := \eta_Q(t) \Lambda_m^{\lambda, \sigma_D}(x, z, t ; y)$. We are going to estimate the right hand side of (3.3.45) using the techniques of Subsection 3.3.2.3. We have

$$
\int H_Q d(\sigma_D - \sigma_Q) = (c_D - c_Q) \int H_Q d\mathcal{H}_D^n + c_Q \int H_Q d(\mathcal{H}_D^n - \mathcal{H}_C^n).
$$

(3.3.46)

We introduce the intermediate v-cubes between $Q \in D_j$ and $D \in \mathcal{D}_m$ to obtain

$$
|c_D - c_Q| \lesssim \alpha(C_b D) + \sum_{R \in \mathcal{D} : Q \subset R \subset C_b D} \alpha(R).
$$

(3.3.47)

Claim 3.3.7. For $x \in D$, $j \geq m - 1$, $|x - y| \lesssim 2^{-m}$, and $z \in \text{supp} \widetilde{\nu}_2^{2-j}(-y)$, the following holds: $|\Lambda_m^{\lambda, \sigma_D}(x, z, t ; y)| \lesssim 2^{m(2n+1) - j}$.

Combining Claim 3.3.7 with (3.3.47), we derive that

$$
|c_D - c_Q| \int |H_Q| d\mathcal{H}_D^n \lesssim 2^{m(2n+1) - j(n+1)} \left( \alpha(C_b D) + \sum_{R \in \mathcal{D} : Q \subset R \subset C_b D} \alpha(R) \right).
$$

(3.3.48)

For the second term on the right side of (3.3.46), one can also use the arguments in Subsection 3.3.2.3 (see (3.3.42) and following) to show that

$$
\left| \int H_Q d(\mathcal{H}_D^n - \mathcal{H}_C^n) \right| \lesssim 2^{m(2n+1) - j(n+1)} \left( \alpha(C_b D) + \sum_{R \in \mathcal{D} : Q \subset R \subset C_b D} \alpha(R) \right)
$$

(3.3.49)

(now it is easier because the function $H_Q(t)$ only depends on the first $n$ coordinates of the points involved, i.e., it depends only on $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ and $\tilde{t}$, so when we project vertically to deal with the image measure, the function $H_Q$ is not affected). Therefore, by (3.3.48), (3.3.49), (3.3.46), and (3.3.45), we obtain

$$
|\Lambda_m^\lambda(x, z ; y) - \Lambda_m^{\sigma_D}(x, z ; y)| \lesssim \sum_{Q \in D_j; Q \subset C_b D} 2^{m(2n+1) - j(n+1)} \left( \alpha(C_b D) + \sum_{R \in \mathcal{D} : Q \subset R \subset C_b D} \alpha(R) \right).
$$
From the definition of $G_3^m_j(x)$ in (3.3.34), we conclude that
\[
|G_3^m_j(x)| \lesssim \sum_{Q \in \mathcal{D}_j : Q \subset C_1 D} 2^{m(2n+1)-j(n+1)} \left( \alpha(C_bD) + \sum_{R \in \mathcal{D}_j : Q \subset R \subset C_2 D} \alpha(R) \right) \int_{Q} \frac{\gamma(z)}{\gamma(y)} |K(z-y)| d\sigma_D(z) d\sigma_D(y)
\]
which, together with the estimate of $|G_3^m_j(x)|$ in Subsection 3.3.2.3, gives (3.3.36).

### 3.3.3 Proof of Claims 3.3.1, ..., 3.3.7

We have to prove:

- **Claim 3.3.1**: For all $x, y, z \in \mathbb{R}^d$, $|\Lambda_m^\mu(x, z; y)| \lesssim 2^{mn}$ and $|\nabla_2 \Lambda_m^\mu(x, z; y)| \lesssim 2^{m(n+1)}$.
- **Claim 3.3.2**: For $j < m$, $y \in \text{supp}\tilde{\tau}^2_{2^{-j-1}}(x-\cdot)$, and $|\tilde{x} - \tilde{z}| \leq 2^{-m}\sqrt{n}$, the following hold: $|\Lambda_m^\mu(x, z; y)| \lesssim 2^{mn}$ and $|\nabla_2 \Lambda_m^\mu(x, z; y)| = 0$.
- **Claim 3.3.3**: We have $|\nabla_2^{\mu,\sigma}(x, z, t; y)| \lesssim 2^{mn}$ and $|\nabla_1 \nabla_2^{\mu,\sigma}(x, z, t; y)| \lesssim 2^{m(2n+1)}$ for all $x, y, z, t \in \mathbb{R}^d$.
- **Claim 3.3.4**: For $z \in \text{supp}\tilde{\tau}^2_{2^{-j-1}}(\cdot - y)$, the following hold: $|\Lambda_m^\mu(x, z; y)| \lesssim 2^{m(n+1)-j}$, $|\nabla_2 \Lambda_m^\mu(x, z; y)| \lesssim 2^{m(n+1)}$, and $|\nabla_1 \Lambda_m^\mu(x, z; y)| \lesssim 2^{m(n+1)}$.
- **Claim 3.3.5**: For $z \in \text{supp}\tilde{\tau}^2_{2^{-j-1}}(\cdot - y)$, $|\Lambda_m^\mu(x, z; y)| \lesssim 2^{m(n+1)-j}$.
- **Claim 3.3.6**: For $x \in D$, $j \geq m-1$, $|x-y| \lesssim 2^{-m}$ and $y \in \text{supp}\tilde{\tau}^2_{2^{-j-1}}(\cdot - y)$, the following hold: $|\Lambda_m^{\mu,\lambda}(x, z, t; y)| \lesssim 2^{m(2n+1)-j}$ and $|\nabla_1 \Lambda_m^{\mu,\lambda}(x, z, t; y)| \lesssim 2^{m(2n+1)}$.
- **Claim 3.3.7**: For $x \in D$, $j \geq m-1$, $|x-y| \lesssim 2^{-m}$ and $y \in \text{supp}\tilde{\tau}^2_{2^{-j-1}}(\cdot - y)$, the following holds: $|\Lambda_m^{\mu,\lambda}(x, z, t; y)| \lesssim 2^{m(2n+1)-j}$.

To prove the claims, we need to express the function $\Lambda$ at the end of Subsection 3.1.3 in a more convenient way. Notice that we can replace $D_m^a$ by $\overline{D}_m^a$ in the definition of $\Lambda$ because $\mu$ and the $n$-dimensional Hausdorff measure vanish on $\partial D_m^a$.

For $u \in \mathbb{R}^n$ and $r > 0$, we denote $|u|_\infty := \max_{i=1,\ldots,n} |u^i|$, $B_\infty(u, r) := \{v \in \mathbb{R}^n : |u - v|_\infty \leq r\}$, and $B_m^\infty(u) := B_\infty(u, 2^{-m-1})$. Given $a \in \mathbb{R}^n$, let $b := a + \{2^{-m-1}\}^n \in \mathbb{R}^n$ be the center of $\bar{D}_m^a$. Then, given $q \in \mathbb{R}^d$,
\[
q \in \overline{D}_m^a \iff |q - b|_\infty \leq 2^{-m} \iff b \in B_m^\infty(q).
\]

Let $\mu_1, \ldots, \mu_k$ be positive Borel measures such that $\mu_l(D_m^a) > 0$ and $\mu_l(\partial D_m^a) = 0$ for all $a \in \mathbb{R}^n$, $m \in \mathbb{Z}$ and $l = 1, \ldots, k$. Given $m \in \mathbb{Z}$ and $x_1, \ldots, x_i, y_1, \ldots, y_j \in \mathbb{R}^d$ we have, by
the change of variable $b = a + \{2^{-m-1}\}^n \in \mathbb{R}^n$,
\[
\Lambda_{m_1,\ldots,m_k}(x_1, \ldots, x_l; y_1, \ldots, y_j) = \int \frac{2^{nm} da}{\prod_{l=1}^{k} \mu(D_m^a)} \left( 2^{nm} \int \chi_{B_m(x) \cap B_m(y)}(b) \, db \right).
\] (3.3.50)

**Proof of Claim 3.3.1.** By (3.3.50), we have
\[
\Lambda_m^y(x, z; y) = 2^{nm} \int \mu(D_m^{b-(2^{-m-1})^n})^{-1} \chi_{B_m(x) \cap B_m(y)}(b) \, db.
\] (3.3.51)

Since $\mu(D_m^b) \geq 2^{-mn}$ for all $b \in \mathbb{R}^n$,
\[
|\Lambda_m^y(x, z; y)| \leq 2^{2mn} \mathcal{L}^n(B_m^y(\tilde{x}) \cap B_m^y(\tilde{z}) \cap B_m^y(\tilde{y})^c) \leq 2^{2mn} \mathcal{L}^n(B_m^y(\tilde{x})) \leq 2^{mn}.
\]

To deal with the second inequality in Claim 3.3.1, we will estimate
\[
|\Lambda_m^y(x, z_1; y) - \Lambda_m^y(x, z_2; y)|/|z_1 - z_2|
\]
for $z_1$ close enough to $z_2$. Recall that, given two sets $F_1, F_2 \subset \mathbb{R}^n$, $F_1 \Delta F_2 \defeq (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$ denotes their symmetric difference. Using (3.3.51), we get
\[
|\Lambda_m^y(x, z_1; y) - \Lambda_m^y(x, z_2; y)| \leq 2^{2mn} \int \chi_{B_m^y(\tilde{x}) \cap B_m^y(\tilde{z_1}) \cap B_m^y(\tilde{y})^c} - \chi_{B_m^y(\tilde{x}) \cap B_m^y(\tilde{z_2}) \cap B_m^y(\tilde{y})^c} \, db
\]
\[
= 2^{2mn} \mathcal{L}^n\left( (B_m^y(\tilde{x}) \cap B_m^y(\tilde{z_1}) \cap B_m^y(\tilde{y})^c) \Delta (B_m^y(\tilde{x}) \cap B_m^y(\tilde{z_2}) \cap B_m^y(\tilde{y})^c) \right)
\]
\[
\leq 2^{2mn} |\tilde{z_1} - \tilde{z_2}|2^{-m(n-1)} \leq 2^{m(n+1)}|z_1 - z_2|,
\] (3.3.52)
and the claim follows. \hfill \Box

**Proof of Claim 3.3.2.** The first estimate has been already proved in Claim 3.3.1. Let us deal with the second one. Notice that if $y \in \text{supp}\tilde{F}_{2^{-j-1}(x-\cdot)}$, then $|\tilde{x} - \tilde{y}| \geq 2^{-j-1}2^{1/n}$. Thus, if also $j < m$ and $|\tilde{x} - \tilde{y}| \leq 2^{-m}2^{-1/n}$, then $|\tilde{x} - \tilde{y}| > 2^{-m}2^{-1/n}$ and $|\tilde{x} - \tilde{y}| > 2^{-m}2^{-1/n}$.

Therefore, $B_m^y(\tilde{x}) \cap B_m^y(\tilde{z}) \cap B_m^y(\tilde{y})^c = B_m^y(\tilde{x}) \cap B_m^y(\tilde{z})$ for all $y \in \text{supp}\tilde{F}_{2^{-j-1}(x-\cdot)}$, if $|\tilde{x} - \tilde{z}| \leq 2^{-m}2^{-1/n}$. This means, using (3.3.51), that $\Lambda_m^y(x, z; y)$ does not depend on $y$, so $\nabla_y \Lambda_m^y(x, z; y) = 0$ for all $y \in \text{supp}\tilde{F}_{2^{-j-1}(x-\cdot)}$, and the claim is proved. \hfill \Box

**Proof of Claim 3.3.3.** This claim follows by arguments very similar to the ones in the proof of Claim 3.3.1. Just notice that $\mu(D_m^b)\sigma_D(D_m^b) \geq 2^{-2mn}$ for all $b \in \mathbb{R}^n$.

**Proof of Claim 3.3.4.** Using (3.3.51), we have that
\[
|\Lambda_m^y(x, z; y)| \leq 2^{2mn} \mathcal{L}^n(B_m^y(\tilde{x}) \cap B_m^y(\tilde{z}) \cap B_m^y(\tilde{y})^c) \leq 2^{2mn} \mathcal{L}^n(B_m^y(\tilde{x}) \cap B_m^y(\tilde{y})^c).
\]
Notice that $\mathcal{L}^n(B^m_\infty(z) \cap B^m_\infty(y)^c) \leq 2^{-m(n-1)}|\tilde{y} - \tilde{z}|$. Since $z \in \text{supp} \tilde{\varphi}_{2^{-j-1}}(\cdot - y)$, $|\tilde{y} - \tilde{z}| \leq 2^{-j}3\sqrt{n}$. Thus, $\mathcal{L}^n(B^m_\infty(z) \cap B^m_\infty(y)^c) \leq 2^{-m(n-1)-j}$, and then $|\Lambda^m_\mu(x, z; y)| \leq 2^{m(n+1)-j}$.

In Claim 3.3.1 we already proved that $|\nabla_z \Lambda^m_\mu(x, z; y)| \leq 2^{m(n+1)}$. Finally, to prove that $|\nabla_y \Lambda^m_\mu(x, z; y)| \leq 2^{m(n+1)}$, one can repeat the computations done in (3.3.52) but applied to the $y$ coordinate and use that $B^m_\infty(\tilde{y}_1)^c \Delta B^m_\infty(\tilde{y}_2)^c = B^m_\infty(\tilde{y}_1) \Delta B^m_\infty(\tilde{y}_2)$.

Proof of Claim 3.3.5. This claim is included in the previous one.

Proof of Claim 3.3.6. Recall that $\lambda = \sum_{Q\in D_j; \sigma \subset C, \cdot \subset D} \eta_Q \sigma_Q$, where $C_\cdot$ is some fixed constant big enough (see the beginning of Subsection 3.3.2). Using the properties of $\eta_Q$ and that $C_\cdot$ is big enough, it is not difficult to show that $\lambda(D^b_m(2^{-m-1})^n) \gtrsim 2^{-mn}$ for all $b \in \mathbb{R}^n$ such that $b \in B^m_\infty(\tilde{x})$ (recall that $x \in D$ and $j \geq m - 1$). Therefore, by (3.3.50),

$|\Lambda^{{\mu}\lambda}_m(x, z, t; y)| \leq 2^{3mn} \mathcal{L}^n(B^m_\infty(\tilde{x}) \cap B^m_\infty(\tilde{z}) \cap B^m_\infty(\tilde{t}) \cap B^m_\infty(\tilde{y})^c) \leq 2^{3mn} \mathcal{L}^n(B^m_\infty(\tilde{z}) \cap B^m_\infty(\tilde{y})^c)$. 

As in the proof of Claim 3.3.4, we have $\mathcal{L}^n(B^m_\infty(\tilde{z}) \cap B^m_\infty(\tilde{y})^c) \approx 2^{-m(n-1)-j}$ for all $z \in \text{supp} \tilde{\varphi}_{2^{-j-1}}(\cdot - y)$. Thus, $|\Lambda^{{\mu}\lambda}_m(x, z, t; y)| \leq 2^{m(2n+1)-j}$, as wished.

For the second estimate in Claim 3.3.6, we argue as in (3.3.52). For $t_1$ and $t_2$ close enough,

$|\Lambda^{{\mu}\lambda}_m(x, z, t_1; y) - \Lambda^{{\mu}\lambda}_m(x, z, t_2; y)|$

$\leq 2^{3mn} \int |\chi_{\mathcal{L}^n(B^m_\infty(\tilde{x}) \cap B^m_\infty(\tilde{z}) \cap B^m_\infty(\tilde{t}) \cap B^m_\infty(\tilde{y})^c)(b)} - \chi_{\mathcal{L}^n(B^m_\infty(\tilde{z}) \cap B^m_\infty(\tilde{t}) \cap B^m_\infty(\tilde{y})^c)(b)}| db$

$\leq 2^{3mn} \mathcal{L}^n(B^m_\infty(\tilde{t}_1)^c \Delta B^m_\infty(\tilde{t}_2)^c) \approx 2^{3mn} |\tilde{t}_1 - \tilde{t}_2| 2^{-m(n-1)} \leq 2^{m(2n+1)}|t_1 - t_2|$, 

and the claim follows by letting $t_1 \to t_2$.

Proof of Claim 3.3.7. This claim is proved as the first estimate in Claim 3.3.6, replacing $\mu$ by $\sigma_D$ (we only used that $\mu(D^b_m)^c \gtrsim 2^{-mn}$ for all $b \in \mathbb{R}^n$, which also holds for $\sigma_D$).

3.4 Proof of Theorem 3.2.2

Given $x \in \Gamma$, let $\{\epsilon_m\}_{m \in \mathbb{Z}}$ be a decreasing sequence of positive numbers such that

$S\mu(x)^2 \leq 2 \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}; \epsilon_m, \epsilon_{m+1} \in I_j} |(K_{\varphi_\epsilon}^{\epsilon_m} * \mu)(x)|^2$,  \hspace{1cm} (3.4.1)

so $\{\epsilon_m\}_{m \in \mathbb{Z}}$ depends on $x$.

Fix $j \in \mathbb{Z}$ and assume that $x \in D$, for some $D \in D_j$. Let $L_D$ be an $n$-plane that minimizes $\alpha(D)$ and let $\sigma_D := c_D \mathcal{H}^n_{L_D}$ be a minimizing measure for $\alpha(D)$. Let $L_D^\parallel$ be the $n$-plane parallel to $L_D$ which contains $x$, and set $\sigma_D^\parallel := c_D \mathcal{H}^n_{L_D^\parallel}$.
3.4. Proof of Theorem 3.2.2

By the antisymmetry of the function $\bar{\varphi}_{\epsilon_{m+1}} K$, and since $\sigma_D^\epsilon$ is a Hausdorff measure on the $n$-plane $L_D^\epsilon$ and $x \in L_D^\epsilon$, we have $(K\bar{\varphi}_{\epsilon_{m+1}}^\epsilon * \sigma_D^\epsilon)(x) = \int \bar{\varphi}_{\epsilon_{m+1}}^\epsilon(x-y)K(x-y)\,d\sigma_D^\epsilon(y) = 0$ for all $m \in \mathbb{Z}$. Therefore, we can decompose

$$(K\bar{\varphi}_{\epsilon_{m+1}}^\epsilon * \mu)(x) = (K\bar{\varphi}_{\epsilon_{m+1}}^\epsilon * (\mu - \sigma_D))(x) + (K\bar{\varphi}_{\epsilon_{m+1}}^\epsilon * (\sigma_D - \sigma_D^\epsilon))(x). \quad (3.4.2)$$

For every $m \in \mathbb{Z}$ such that $\epsilon_m, \epsilon_{m+1} \in I_j$ we will prove the following inequalities:

$$|(K\bar{\varphi}_{\epsilon_{m+1}}^\epsilon * (\mu - \sigma_D))(x)| \lesssim 2^j|\epsilon_m - \epsilon_{m+1}|\alpha(D), \quad (3.4.3)$$

$$|(K\bar{\varphi}_{\epsilon_{m+1}}^\epsilon * (\sigma_D - \sigma_D^\epsilon))(x)| \lesssim 2^{2j}|\epsilon_m - \epsilon_{m+1}|\text{dist}(x, L_D). \quad (3.4.4)$$

Assume for a moment that these estimates hold. Then, by (3.4.2),

$$|(K\bar{\varphi}_{\epsilon_{m+1}}^\epsilon * \mu)(x)| \lesssim 2^j|\epsilon_m - \epsilon_{m+1}|\left(\alpha(D) + 2^{j}\text{dist}(x, L_D)\right).$$

Then, using (3.4.1), we conclude that

$$\|S\mu\|_{L^2(\mu)}^2 \leq 2 \sum_{j \in \mathbb{Z}} \sum_{D \in D_j} \int_{D} \sum_{m \in \mathbb{Z}, \epsilon_m, \epsilon_{m+1} \in I_j} |(K\bar{\varphi}_{\epsilon_{m+1}}^\epsilon * \mu)(x)|^2 \,d\mu(x) \leq 2 \sum_{j \in \mathbb{Z}} \sum_{D \in D_j} \int_D \sum_{m \in \mathbb{Z}, \epsilon_m, \epsilon_{m+1} \in I_j} \left(\frac{|\epsilon_m - \epsilon_{m+1}|}{2^{-j}}\right)^2 \,d\mu(x) \lesssim \sum_{D \in D} \alpha(D)^2 \mu(D) + \sum_{D \in D} \int_D \left(\frac{\text{dist}(x, L_D)}{\ell(D)}\right)^2 \,d\mu(x).$$

Notice that, under the integral sign on the right hand side of the first inequality above, $\epsilon_m$ and $\epsilon_{m+1}$ depend on $x$. It is not obvious that the resulting function is measurable. To deal with this issue more carefully, we might first ask $\{\epsilon_m\}_{m \in \mathbb{Z}}$ to lie in some finite set, prove the variational norm inequality with constants independent of the set, and then enlarge the set. Then, by monotone convergence we would obtain the result with $\{\epsilon_m\}_{m \in \mathbb{Z}}$ restricted to a countable set dense in $(0, \infty)$. The final theorem would follow then from the continuity properties of the operators involved. This applies to other similar situations in the chapter. However, for the sake of conciseness, we will skip further details.

The second term on the right hand side of the last inequality coincides with $W_1\mu$ (see (3.3.6)), thus it is bounded (modulo constants) by $\sum_{D \in D} (\alpha(D) + \beta_2(D)\mu(D))$, and Theorem 3.2.2 is proved.

It only remains to verify (3.4.3) and (3.4.4) for $x \in D \in D_j$ and $m \in \mathbb{Z}$ such that $\epsilon_m, \epsilon_{m+1} \in I_j$. First of all, notice that $\bar{\varphi}_{\epsilon_{m+1}}^\epsilon$ satisfies

$$|\bar{\varphi}_{\epsilon_{m+1}}^\epsilon(x-y)| = |\varphi^{\epsilon_m}_{\epsilon_m} \left(\frac{x-y}{\epsilon_{m+1}}\right) - \varphi^{\epsilon_m}_{\epsilon_m} \left(\frac{x-y}{\epsilon_m}\right)| \leq \|\varphi^{\epsilon_m}_{\epsilon_m}\|_{L^\infty(\mathbb{R})} \left|\frac{x-y}{\epsilon_{m+1}}\right| - \frac{|x-y|}{\epsilon_m} \lesssim 2^{j}|\epsilon_m - \epsilon_{m+1}| \quad (3.4.5)$$

where
for all \( y \in \text{supp} \tilde{\varphi}_{\epsilon_{m+1}}^\epsilon(x - \cdot) \). For \( i = 1, \ldots, d \),

\[
\partial_{g^i} \left( \tilde{\varphi}_m(x - y) \right) = \varphi_R \left( \frac{|\tilde{x} - y|}{\epsilon_m} \right) \frac{g^i - x^i}{\epsilon_m |\tilde{x} - y|} \chi_{[1,n]}(i).
\]

Hence,

\[
|\partial_{g^i} \left( \tilde{\varphi}_{\epsilon_{m+1}}^\epsilon(x - y) \right)| \leq \left| \varphi_R \left( \frac{|\tilde{x} - y|}{\epsilon_m} \right) \frac{1}{\epsilon_m} - \varphi_R \left( \frac{|\tilde{x} - y|}{\epsilon_{m+1}} \right) \frac{1}{\epsilon_{m+1}} \right|
\]

\[
\leq \left| \varphi_R' \left( \frac{|\tilde{x} - y|}{\epsilon_m} \right) \frac{1}{\epsilon_m} - \frac{1}{\epsilon_{m+1}} \right| + \left| \varphi_R' \left( \frac{|\tilde{x} - y|}{\epsilon_m} \right) - \varphi_R' \left( \frac{|\tilde{x} - y|}{\epsilon_{m+1}} \right) \right| \frac{1}{\epsilon_{m+1}}
\]

\[
\leq \left( \|\varphi_R'\|_\infty + \|\varphi_R''\|_\infty \right) \frac{\epsilon_m - \epsilon_{m+1}}{\epsilon_m \epsilon_{m+1}}.
\]

Since \( \epsilon_m, \epsilon_{m+1} \in I_j \), we deduce from the previous estimate that, for \( x - y \in \text{supp} \tilde{\varphi}_{\epsilon_{m+1}}^\epsilon \),

\[
|\nabla_y \left( \tilde{\varphi}_{\epsilon_{m+1}}^\epsilon(x - y) \right)| \lesssim \frac{\epsilon_m - \epsilon_{m+1}}{\epsilon_m \epsilon_{m+1}} \approx 2^j |\epsilon_m - \epsilon_{m+1}|.
\]

(3.4.6)

We are going to use (3.4.5) and (3.4.6) to prove (3.4.3) and (3.4.4). Let us start with (3.4.3). Since \( \epsilon_m, \epsilon_{m+1} \in I_j \), we can assume that \( \tilde{\varphi}_{\epsilon_{m+1}}^\epsilon(x - \cdot) \subset B_D \), by taking \( C_\Gamma \) big enough.

By (3.4.5) and (3.4.6), for all \( y \in \text{supp} \tilde{\varphi}_{\epsilon_{m+1}}^\epsilon(x - \cdot) \),

\[
|\nabla_y \left( \tilde{\varphi}_{\epsilon_{m+1}}^\epsilon(x - y) K(x - y) \right)| \lesssim 2^j |\epsilon_m - \epsilon_{m+1}|,
\]

hence \(|(K\tilde{\varphi}_{\epsilon_{m+1}}^\epsilon \ast (\mu - \sigma_D))(x)| \lesssim 2^j |\epsilon_m - \epsilon_{m+1}||\text{dist}_{B_D}(\mu, \sigma_D) \lesssim 2^j |\epsilon_m - \epsilon_{m+1}||\alpha(D) \), which gives (3.4.3).

In order to prove (3.4.4), set \( L_D^x = \{ (\tilde{t}, a(\tilde{t})) \in \mathbb{R}^d : \tilde{t} \in \mathbb{R}^n \} \), where \( a : \mathbb{R}^n \to \mathbb{R}^{d-n} \) is an appropriate affine map, and let \( p : L_D \to L_D^x \) be the map defined by \( p(t) := (\tilde{t}, a(\tilde{t})) \). Since \( \Gamma \) is a Lipschitz graph, \( a \) is well defined and \( p \) is a homeomorphism. Let \( p_\ast \mathcal{H}^n_{L_D} \) be the image measure of \( \mathcal{H}^n_{L_D} \) by \( p \). It is easy to see that, \(|y - p(y)| \approx \text{dist}(x, L_D) \) for all \( y \in L_D \). Notice also that the image measure \( p_\ast \mathcal{H}^n_{L_D} \) coincides with \( \mathcal{H}^n_{L_D} \). Therefore, since \( \tilde{\varphi}_{\epsilon_{m+1}}^\epsilon(x - y) \) only depends on \( \tilde{x} - \tilde{y} \),

\[
(K\tilde{\varphi}_{\epsilon_{m+1}}^\epsilon \ast (\sigma_D - \sigma_D^\epsilon))(x) = C_D \int \tilde{\varphi}_{\epsilon_{m+1}}^\epsilon(x - y) K(x - y) d(\mathcal{H}^n_{L_D} - p_\ast \mathcal{H}^n_{L_D})(y)
\]

\[
= C_D \int \tilde{\varphi}_{\epsilon_{m+1}}^\epsilon(x - y)(K(x - y) - K(x - p(y))) d\mathcal{H}^n_{L_D}(y).
\]

(3.4.7)

For \( y \in \text{supp} \tilde{\varphi}_{\epsilon_{m+1}}^\epsilon(x - \cdot) \cap L_D \), we have

\[
|K(x - y) - K(x - p(y))| \lesssim 2^{j(n+1)}|y - p(y)| \approx 2^{j(n+1)} \text{dist}(x, L_D).
\]

Plugging this estimate and (3.4.5) into (3.4.7), we conclude that

\[
|(K\tilde{\varphi}_{\epsilon_{m+1}}^\epsilon \ast (\sigma_D - \sigma_D^\epsilon))(x)| \lesssim 2^{2j} |\epsilon_m - \epsilon_{m+1}||\text{dist}(x, L_D),
\]

which gives (3.4.4); and the theorem follows.
3.5 \( L^2 \) localization of \( \mathcal{V}_\rho \circ \mathcal{T}^{H^n_\Gamma}_{\tilde{\varphi}} \) and \( \mathcal{O} \circ \mathcal{T}^{H^n_\Gamma}_{\tilde{\varphi}} \)

From here till the end of the chapter, \( \Gamma := \{ x \in \mathbb{R}^d : x = (\tilde{x}, A(\tilde{x})) \} \) will be the graph of a Lipschitz function \( A : \mathbb{R}^n \to \mathbb{R}^{d-n} \), without any assumption on the support of \( A \).

**Theorem 3.5.1.** Let \( \rho > 2 \). There exist \( C_1, C_2 > 0 \) such that, for every \( f \in L^\infty(\mathcal{H}^n_\Gamma) \) supported in \( \Gamma \cap D \) (where \( D := \tilde{D} \times \mathbb{R}^{d-n} \) and \( \tilde{D} \) is a cube of \( \mathbb{R}^n \)),

\[
\int_D \left( (\mathcal{V}_\rho \circ \mathcal{T}^{H^n_\Gamma}_{\tilde{\varphi}}) f \right)^2 d\mathcal{H}^n_\Gamma \leq C_1 \| f \|_{L^\infty(\mathcal{H}^n_\Gamma)}^2 \mathcal{H}_\Gamma^n(D) \quad \text{and} \quad (3.5.1)
\]

\[
\int_D \left( (\mathcal{O} \circ \mathcal{T}^{H^n_\Gamma}_{\tilde{\varphi}}) f \right)^2 d\mathcal{H}^n_\Gamma \leq C_2 \| f \|_{L^\infty(\mathcal{H}^n_\Gamma)}^2 \mathcal{H}_\Gamma^n(D). \quad (3.5.2)
\]

The constant \( C_2 \) does not depend on the fixed sequence that defines \( \mathcal{O} \).

We will only give the proof of (3.5.1), because the proof of (3.5.2) follows by very similar arguments and computations.

We claim that it is enough to prove (3.5.1) for all functions \( f \) such that \( f(x) \approx 1 \) for all \( x \in \Gamma \cap D \). Otherwise, we consider \( g(x) := \| f \|_{L^\infty(\mathcal{H}^n_\Gamma)}^{-1} f(x) + 2\chi_D(x) \), which clearly satisfies \( g(x) \approx 1 \) for all \( x \in \Gamma \cap D \). Since \( f = \| f \|_{L^\infty(\mathcal{H}^n_\Gamma)} (g - 2\chi_D) \),

\[
(\mathcal{V}_\rho \circ \mathcal{T}^{H^n_\Gamma}_{\tilde{\varphi}}) f(x) \leq \| f \|_{L^\infty(\mathcal{H}^n_\Gamma)} (\mathcal{V}_\rho \circ \mathcal{T}^{H^n_\Gamma}_{\tilde{\varphi}}) g(x) + 2(\mathcal{V}_\rho \circ \mathcal{T}^{H^n_\Gamma}_{\tilde{\varphi}}) \chi_D(x)).
\]

Applying (3.5.1) to the functions \( g \) and \( \chi_D \), we get \( \int_D \left( (\mathcal{V}_\rho \circ \mathcal{T}^{H^n_\Gamma}_{\tilde{\varphi}}) f \right)^2 d\mathcal{H}^n_\Gamma \lesssim \| f \|_{L^\infty(\mathcal{H}^n_\Gamma)}^2 \mathcal{H}_\Gamma^n(D) \).

Given \( f \) and \( D \) as in Theorem 3.5.1, from now on, we assume that \( f \approx 1 \) in \( \Gamma \cap D \). Let \( \tilde{z}_D \) be the center of \( \tilde{D} \) and set \( z_D := (\tilde{z}_D, A(\tilde{z}_D)) \). One can easily construct a Lipschitz function \( A_D : \mathbb{R}^n \to \mathbb{R}^{d-n} \) such that Lip(\( A_D \)) \( \lesssim \) Lip(\( A \)), \( A_D(\tilde{x}) = A(\tilde{z}_D) \) for all \( \tilde{x} \in (3\tilde{D})^c \), and \( A_D(\tilde{x}) = A(\tilde{x}) \) for all \( \tilde{x} \in \tilde{D} \). Let \( \Gamma_D \) be the graph of \( A_D \) and define the measure \( \mu := \mathcal{H}^n_{\Gamma_D \setminus D} + f \mathcal{H}^n_{\Gamma_D \setminus D} \). Notice that \( \chi_{(3D)^c} \mu \) is supported in the \( n \)-plane \( L := \mathbb{R}^n \times \{ A(\tilde{z}_D) \} \) and \( \chi_{D} \mu = f \mathcal{H}^n_{\Gamma_D} \).

Since \( f \approx 1 \) in \( \Gamma \cap D \) and \( \chi_D \mu = (1 - \chi_{(3D)^c} - \chi_{3D \setminus D}) \mu \), we can decompose

\[
\int_D \left( (\mathcal{V}_\rho \circ \mathcal{T}^{H^n_\Gamma}_{\tilde{\varphi}}) f \right)^2 d\mathcal{H}^n_\Gamma \approx \int_D \mathcal{V}_\rho(K\tilde{\varphi} \ast (\chi_D \mu))^2 d\mu \\
\lesssim \int_D \left( \mathcal{V}_\rho(K\tilde{\varphi} \ast \mu) + \mathcal{V}_\rho(K\tilde{\varphi} \ast (\chi_{(3D)^c} \mu)) + \mathcal{V}_\rho(K\tilde{\varphi} \ast (\chi_{3D \setminus D} \mu)) \right)^2 d\mu.
\]

In the next subsections, we will see that \( \int_D \mathcal{V}_\rho(K\tilde{\varphi} \ast \mu)^2 d\mu \), \( \int_D \mathcal{V}_\rho(K\tilde{\varphi} \ast (\chi_{(3D)^c} \mu))^2 d\mu \), and \( \int_D \mathcal{V}_\rho(K\tilde{\varphi} \ast (\chi_{3D \setminus D} \mu))^2 d\mu \) are bounded by \( C\mu(D) \), and (3.5.1) will follow.
3.5.1 Proof of $\int_D V_\rho(K\tilde{\varphi} \ast \mu)^2 \, d\mu \lesssim \mu(D)$

Fix $x \in \text{supp}\mu$, and let $\{\epsilon_m\}_{m \in \mathbb{Z}}$ be a decreasing sequence of positive numbers (which depends on $x$) such that

$$
(V_\rho(K\tilde{\varphi} \ast \mu)(x))^\rho \leq 2 \sum_{m \in \mathbb{Z}} |(K\tilde{\varphi}_{\epsilon_{m+1}} \ast \mu)(x)|^\rho.
$$

(3.5.3)

For $j \in \mathbb{Z}$ we set $I_j := [2^{-j-1}, 2^{-j})$. We decompose $\mathbb{Z} = S \cup L$, where

$$
S := \bigcup_{j \in \mathbb{Z}} S_j, \quad S_j := \{m \in \mathbb{Z} : \epsilon_m, \epsilon_{m+1} \in I_j\},
$$

(3.5.4)

$$
L := \{m \in \mathbb{Z} : \epsilon_m \in I_i, \epsilon_{m+1} \in I_j \text{ for } i < j\}.
$$

Then, $\sum_{m \in \mathbb{Z}} |(K\tilde{\varphi}_{\epsilon_{m+1}} \ast \mu)(x)|^\rho = \sum_{m \in S} |(K\tilde{\varphi}_{\epsilon_{m+1}} \ast \mu)(x)|^\rho + \sum_{m \in L} |(K\tilde{\varphi}_{\epsilon_{m+1}} \ast \mu)(x)|^\rho$.

Notice that, since the $\ell^\rho(\mathbb{Z})$-norm is smaller than the $\ell^2(\mathbb{Z})$-norm for $\rho > 2$,

$$
\sum_{m \in S} |(K\tilde{\varphi}_{\epsilon_{m+1}} \ast \mu)(x)|^\rho \leq S \mu(x)^\rho,
$$

(3.5.5)

where $S \mu(x)$ has been defined in Theorem 3.2.2.

Let us now estimate the sum over the indices $m \in L$. For $m \in \mathbb{Z}$ we define $j(\epsilon_m)$ as the integer such that $\epsilon_m \in I_{j(\epsilon_m)}$. Since $\{\epsilon_m\}_{m \in \mathbb{Z}}$ is decreasing, given $j \in \mathbb{Z}$, there is at most one index $m \in L$ such that $\epsilon_m \in I_j$. Thus, if $k, m \in L$ and $k < m$, one has $j(\epsilon_k) < j(\epsilon_m)$.

With this notation, we have

$$
\sum_{m \in L} |(K\tilde{\varphi}_{\epsilon_{m+1}} \ast \mu)(x)|^\rho = \sum_{m \in L} |(K\tilde{\varphi}_{\epsilon_{m+1}} \ast \mu)(x) - (K\tilde{\varphi}_{\epsilon_m} \ast \mu)(x)|^\rho
\lesssim \sum_{m \in L} |(K\tilde{\varphi}_{\epsilon_{m+1}} \ast \mu)(x) - (K\tilde{\varphi}_{j(\epsilon_{m+1})-1} \ast \mu)(x)|^\rho
+ \sum_{m \in L} |(K\tilde{\varphi}_{j(\epsilon_{m+1})-1} \ast \mu)(x) - E_{j(\epsilon_m)+1}(\mu)(x)|^\rho
+ \sum_{m \in L} |E_{j(\epsilon_m)+1}(\mu)(x) - E_{j(\epsilon_m)+1}(\mu)(x)|^\rho
+ \sum_{m \in L} |E_{j(\epsilon_m)+1}(\mu)(x) - (K\tilde{\varphi}_{2^{-j(\epsilon_m)+1}} \ast \mu)(x)|^\rho
+ \sum_{m \in L} |(K\tilde{\varphi}_{2^{-j(\epsilon_m)+1}} \ast \mu)(x) - (K\tilde{\varphi}_{\epsilon_m} \ast \mu)(x)|^\rho
\lesssim S \mu(x)^\rho + W(\mu(x))^\rho + V_\rho(E(\mu)(x))^\rho,
$$

(3.5.6)

where $S \mu(x)$ and $W(\mu(x))$ have been defined in Theorems 3.2.2 and 3.2.1, respectively, and $V_\rho(E(\mu))$ is the $\rho$-variation of the average of martingales $\{E_m(\mu)\}_{m \in \mathbb{Z}}$ from Subsection 3.1.3. Therefore, by (3.5.5), (3.5.6), and (3.5.3), we deduce that $V_\rho(K\tilde{\varphi} \ast \mu)(x) \lesssim S \mu(x) + W(\mu(x)) + V_\rho(E(\mu)(x))$ for all $x \in \text{supp}\mu$, and so

$$
\int_D V_\rho(K\tilde{\varphi} \ast \mu)^2 \, d\mu \lesssim \|S \mu\|^2_{L^2(\mu)} + \|W(\mu)\|^2_{L^2(\mu)} + \|V_\rho(E(\mu))\|^2_{L^2(\mu)}.
$$

(3.5.7)
3.5. $L^2$ localization of $\mathcal{V}_\rho \circ T_\mathcal{F}^{H_n}$ and $\mathcal{O} \circ T_\mathcal{F}^{H_n}$

Clearly, Theorem 3.1.7, Theorem 3.2.1, and Theorem 3.2.2 can be applied to the measure $\mu$, because $\text{supp}_\mu$ is a translation of the graph of a Lipschitz function with compact support. These theorems in combination with (3.5.7) yield

$$
\int_D \mathcal{V}_\rho(K\widetilde{\varphi} \ast \mu)^2 d\mu \leq C_1(\mu(3D) + \sum_{Q \in D} (\alpha_\mu(C_2Q)^2 + \beta_{2,\mu}(Q)^2)\mu(Q)).
$$

(3.5.8)

where $C_1, C_2 > 0$ only depend on $n, d, K, \text{Lip}(A)$, and $\rho$ (the condition $\rho > 2$ is used to ensure the $L^2$ boundedness of $\mathcal{V}_\rho(E\mu)$). Obviously, $\mu(3D) \approx \mu(D)$ and, since $\chi_{(3D)^c} \mu$ coincides with the $n$-dimensional Hausdorff measure on an $n$-plane, using Remark 3.1.2 it is easy to check that $\sum_{Q \in D} (\alpha_\mu(C_2Q)^2 + \beta_{2,\mu}(Q)^2)\mu(Q) \lesssim \mu(3D)$. Hence, we conclude that $\int_D \mathcal{V}_\rho(K\widetilde{\varphi} \ast \mu)^2 d\mu \lesssim \mu(D)$ by (3.5.8).

3.5.2 Proof of $\int_D \mathcal{V}_\rho(K\widetilde{\varphi} \ast (\chi_{(3D)^c} \mu))^2 d\mu \lesssim \mu(D)$

Fix $x \in \text{supp}_\mu \cap D$, and let $\{\epsilon_m\}_{m \in \mathbb{Z}}$ be a decreasing sequence of positive numbers (which depends on $x$) such that

$$
(\mathcal{V}_\rho(K\widetilde{\varphi} \ast (\chi_{(3D)^c} \mu))(x))^2 \leq 2 \sum_{m \in \mathbb{Z}} |(K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_m+1} \ast (\chi_{(3D)^c} \mu))(x)|^2.
$$

(3.5.9)

Recall that $\bar{z}_D$ is the center of $\bar{D}$, $z_D := (\bar{z}_D, A(\bar{z}_D))$ and $L := \mathbb{R}^n \times \{A(\bar{z}_D)\}$. Since $\chi_{(3D)^c} \mu = H_n^{\mathcal{F}_{\epsilon_m}}$ and $z_D$ is the center of $L \cap D$, $(K\theta_\epsilon \ast (\chi_{(3D)^c} \mu))(z_D) = 0$ for all $0 < \epsilon \leq \delta$. Thus, $|K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_m+1} \ast (\chi_{(3D)^c} \mu))(x)| = |K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_m+1} \ast (\chi_{(3D)^c} \mu))(x) - (K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_m+1} \ast (\chi_{(3D)^c} \mu))(z_D)| \leq \Theta_{1,m} + \Theta_{2,m}$, where

$$
\Theta_{1,m} := \int_{(3D)^c} (K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_m+1}(x-y) - K(x-y) \ast (\chi_{(3D)^c} \mu))(y) d\mu(y),
$$

$$
\Theta_{2,m} := \int_{(3D)^c} (K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_m+1}(x-y) \ast (\chi_{(3D)^c} \mu))(y) - K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_m+1}(z_D-y) \ast (\chi_{(3D)^c} \mu))(z_D-y) d\mu(y).
$$

(3.5.10)

Since $x \in \text{supp}_\mu \cap D$ and $A$ is a Lipschitz function, we have $|x - z_D| \lesssim \ell(D)$, and then $|K(x-y) - K(z_D-y)| \lesssim |x-z_D||z_D-y|^{-n-1} \lesssim \ell(D)|z_D-y|^{-n-1}$ for all $y \in (3D)^c$. Therefore, using that $\sum_{m \in \mathbb{Z}} \widetilde{\varphi}^{\epsilon_m}_{\epsilon_m+1} \leq 1$ and that $\rho > 1$,

$$
\left(\sum_{m \in \mathbb{Z}} \Theta_{1,m}^\rho\right)^{1/\rho} \leq \sum_{m \in \mathbb{Z}} \Theta_{1,m} \lesssim \int_{(3D)^c} \ell(D)|z_D-y|^{-n-1} d\mu(y) \lesssim 1.
$$

(3.5.11)

To deal with $\Theta_{2,m}$, we decompose $\mathbb{Z} = \mathcal{S} \cup \mathcal{L}$ as in (3.5.4). As before, given $m \in \mathbb{Z}$, let $j(\epsilon_m)$ be the integer such that $\epsilon_m \in I_{j(\epsilon_m)}$. Observe that

$$
\text{supp} \widetilde{\varphi}^{\epsilon_m}_{\epsilon_m+1}(x-) \subset A(\bar{x}, 2\sqrt{n}2^{-j(\epsilon_m+1)}1, 3\sqrt{n}2^{-j(\epsilon_m)}) \times \mathbb{R}^{d-n} =: A_m(x).
$$
Notice also that the sets $A_m(x)$ have finite overlap for $m \in \mathcal{L}$, and the same is true for the sets $A'_j(x) := A(\bar{x}, 2.1\sqrt{n2^{-j-1}}, 3.\sqrt{n2^{-j}}) \times \mathbb{R}^{d-n}$ for $j \in \mathbb{Z}$. The same observations hold if we replace $x$ by $z_D$ (and $\bar{x}$ by $\tilde{z}_D$). Obviously, $A_m(x) \subset A'_j(x)$ (and $A_m(z_D) \subset A'_j(z_D)$) for $m \in \mathcal{S}_j$. Assume that $m \in \mathcal{S}$. With the same computations as those carried out in (3.4.6), one can easily prove that, for all $z - y \in \text{supp} \varphi_{\epsilon_{m+1}}$:

$$|\nabla_z(\varphi_{\epsilon_{m+1}}(z - y))| \lesssim \left(\|\varphi'_{\epsilon_{m+1}}\|_{L^\infty(\mathbb{R})} + \|\varphi''_{\epsilon_{m+1}}\|_{L^\infty(\mathbb{R})}\right) \frac{|z - y|}{\epsilon_{m+1}} \lesssim 2^{j(\epsilon_m)} \frac{|\epsilon_m - \epsilon_{m+1}|}{|z - y|},$$

because $|z - y| \approx |\epsilon_m| \approx 2^{j(\epsilon_m)}$ for all $z - y \in \text{supp} \varphi_{\epsilon_{m+1}}$ and $m \in \mathcal{S}$. In particular, if $z \in D$ and $y \in (3D)^c$, $|\nabla_z(\varphi_{\epsilon_{m+1}}(z - y))| \lesssim 2^{j(\epsilon_m)}|\epsilon_m - \epsilon_{m+1}| |z_D - y|^{-1}$. Hence,

$$\Theta_{2m} \lesssim \int_{(A_m(x) \cup A_m(z_D)) \setminus 3D} \ell(D) 2^{j(\epsilon_m)} \frac{|\epsilon_m - \epsilon_{m+1}|}{|z_D - y|^{n+1}} d\mu(y),$$

and then,

$$\left(\sum_{m \in \mathcal{S}} \Theta_{2m}^{\rho}\right)^{1/\rho} \lesssim \sum_{m \in \mathcal{S}} \int_{(A_m(x) \cup A_m(z_D)) \setminus 3D} \ell(D) 2^{j(\epsilon_m)} \frac{|\epsilon_m - \epsilon_{m+1}|}{|z_D - y|^{n+1}} d\mu(y)
\lesssim \sum_{j \in \mathbb{Z}} \int_{(A'_j(x) \cup A'_j(z_D)) \setminus 3D} \ell(D) \frac{|\epsilon_m - \epsilon_{m+1}|}{2^{-j}} d\mu(y) \lesssim 1 \quad (3.5.12)$$

Assume now that $m \in \mathcal{L}$. It is easy to check that $|\nabla_z(\varphi_{\epsilon_{m+1}}(z - y))| \lesssim |z - y|^{-1}$ for all $z, y \in \mathbb{R}^d$. So, if also $z \in D$ and $y \in (3D)^c$, $|\nabla_z(\varphi_{\epsilon_{m+1}}(z - y))| \lesssim |z_D - y|^{-1}$. Therefore,

$$\left(\sum_{m \in \mathcal{L}} \Theta_{2m}^{\rho}\right)^{1/\rho} \lesssim \sum_{m \in \mathcal{L}} \int_{(A_m(x) \cup A_m(z_D)) \setminus 3D} \ell(D) \frac{|\epsilon_m - \epsilon_{m+1}|}{|z_D - y|^{n+1}} d\mu(y) \lesssim \int_{(3D)^c} \frac{\ell(D)}{|z_D - y|^{n+1}} d\mu(y) \lesssim 1 \quad (3.5.13)$$

Finally combining (3.5.11), (3.5.12), and (3.5.13), with (3.5.9) and the fact that $(K \varphi_{\epsilon_{m+1}} * (\chi_{(3D)^c} \mu))(x) \leq \Theta_{1m} + \Theta_{2m}$, we conclude that

$$\nu_\rho(K \varphi * (\chi_{(3D)^c} \mu))(x) \lesssim \left(\sum_{m \in \mathbb{Z}} \Theta_{1m}^{\rho}\right)^{1/\rho} + \left(\sum_{m \in \mathcal{S}} \Theta_{2m}^{\rho}\right)^{1/\rho} + \left(\sum_{m \in \mathcal{L}} \Theta_{2m}^{\rho}\right)^{1/\rho} \lesssim 1$$

for all $x \in \text{supp} \mu \cap D$. Therefore, $\int_D \nu_\rho(K \varphi * (\chi_{(3D)^c} \mu))^2 d\mu \lesssim \mu(D)$. 
3.5.3 Proof of \( \int_D V_\rho(K\tilde{\varphi} * (\chi_{3D\setminus D\mu}))^2 \, d\mu \lesssim \mu(D) \)

Fix \( x \in \text{supp}\mu \cap D \). Since \( \rho > 1 \),

\[
V_\rho(K\tilde{\varphi} * (\chi_{3D\setminus D\mu}))(x) = \sup_{\{\epsilon_m\}} \left( \sum_{m \in \mathbb{Z}} \left| \int_{3D \setminus D} \tilde{\varphi}_{\epsilon_{m+1}}(x-y)K(x-y) \, d\mu(y) \right|^\rho \right)^{1/\rho} \\
\leq \sup_{\{\epsilon_m\}} \sum_{m \in \mathbb{Z}} \int_{3D \setminus D} \tilde{\varphi}_{\epsilon_{m+1}}(x-y) |K(x-y)| \, d\mu(y) \\
\leq \int_{3D \setminus D} |K(x-y)| \, d\mu(y).
\]

By a standard computation, one can show that \( \int_D \left( \int_{3D \setminus D} |K(x-y)| \, d\mu(y) \right)^2 \, d\mu(x) \lesssim \mu(D) \), hence we conclude that \( \int_D V_\rho(K\tilde{\varphi} * (\chi_{3D\setminus D\mu}))^2 \, d\mu \lesssim \mu(D) \).

3.6 \( L^p \) and endpoint estimates for \( V_\rho \circ \mathcal{T}^H_{\tilde{\varphi}} \) and \( \mathcal{O} \circ \mathcal{T}^H_{\tilde{\varphi}} \)

We denote by \( H^1(\mathcal{H}_T^n) \) and \( \text{BMO}(\mathcal{H}_T^n) \) the (atomic) Hardy space and the space of functions with bounded mean oscillation, respectively, with respect to the measure \( \mathcal{H}_T^n \). These spaces are defined as the classical \( H^1(\mathbb{R}^d) \) and \( \text{BMO}(\mathbb{R}^d) \) (see [Du, Chapter 6], for example), but by replacing the true cubes of \( \mathbb{R}^d \) by our special \( v \)-cubes.

**Theorem 3.6.1.** Let \( \rho > 2 \). The operators \( V_\rho \circ \mathcal{T}^H_{\tilde{\varphi}} \) and \( \mathcal{O} \circ \mathcal{T}^H_{\tilde{\varphi}} \) are bounded

- in \( L^p(\mathcal{H}_T^n) \) for \( 1 < p < \infty \),
- from \( H^1(\mathcal{H}_T^n) \) to \( L^1(\mathcal{H}_T^n) \), and
- from \( L^\infty(\mathcal{H}_T^n) \) to \( \text{BMO}(\mathcal{H}_T^n) \),

and the norm of \( \mathcal{O} \circ \mathcal{T}^H_{\tilde{\varphi}} \) in the cases above is bounded independently of the sequence that defines the oscillation.

We will only give the proof of Theorem 3.6.1 in the case of the \( \rho \)-variation, because the proof for the oscillation follows by analogous arguments.

3.6.1 The operator \( V_\rho \circ \mathcal{T}^H_{\tilde{\varphi}} : H^1(\mathcal{H}_T^n) \to L^1(\mathcal{H}_T^n) \) is bounded

Fix a cube \( \tilde{D} \subset \mathbb{R}^n \) and set \( D := \tilde{D} \times \mathbb{R}^{d-n} \). Let \( f \) be an atom, i.e., a function defined on \( \Gamma \) and such that

\[
\text{supp} f \subset D, \quad \|f\|_{L^\infty(\mathcal{H}_T^n)} \leq \frac{1}{\mathcal{H}_T^n(D)}, \quad \text{and} \quad \int f \, d\mathcal{H}_T^n = 0.
\]  

We have to prove that \( \int (V_\rho \circ \mathcal{T}^H_{\tilde{\varphi}}) f \, d\mathcal{H}_T^n \leq C \), for some constant \( C > 0 \) which does not depend on \( f \) or \( D \). Since \( (V_\rho \circ \mathcal{T}^H_{\tilde{\varphi}}) f(x) \) is well defined and non negative for \( f \in L^1(\mathcal{H}_T^n) \), the
uniform boundedness of \( V_\rho \circ T^{H^n}_\varphi \) on atoms yields its boundedness from \( H^1(\mathcal{H}^n) \) to \( L^1(\mathcal{H}^n) \) by standard arguments. We omit the details.

First of all, by Hölder’s inequality, Theorem 3.5.1, and (3.6.1),

\[
\int_{3D} (V_\rho \circ T^{H^n}_\varphi) f \, d\mathcal{H}^n \leq \mathcal{H}^n(3D)^{1/2} \left( \int_{3D} ((V_\rho \circ T^{H^n}_\varphi) f)^2 \, d\mathcal{H}^n \right)^{1/2} \lesssim \mathcal{H}^n(3D)^{1/2} \left( \|f\|_{L^\infty(\mathcal{H}^n)}^2 \mathcal{H}^n(3D) \right)^{1/2} \lesssim 1.
\]

Thus, it remains to prove that \( \int_{(3D)\setminus(V_\rho \circ T^{H^n}_\varphi)} f \, d\mathcal{H}^n \leq C \).

Given \( x \in \Gamma \setminus 3D \), let \( \{\epsilon_m\}_{m \in \mathbb{Z}} \) be a decreasing sequence of positive numbers (which depends on \( x \)) such that

\[
((V_\rho \circ T^{H^n}_\varphi) f(x))^p \leq 2 \sum_{m \in J} |(K\varphi_{\epsilon_{m+1}} \ast (f\mathcal{H}^n))(x)|^p,
\]

where \( J := \{m \in \mathbb{Z} : \text{supp } \varphi_{\epsilon_{m+1}}(x - \cdot) \cap \text{supp } f \neq \emptyset\} \), thus \( \{\epsilon_m\}_{m \in \mathbb{Z}} \) depends on \( x \).

Set \( z_D := (\tilde{z}_D, A(\tilde{z}_D)) \in D \cap \Gamma \), where \( \tilde{z}_D \) is the center of \( \tilde{D} \). By (3.6.1), we have

\[
\int \varphi_{\epsilon}(x - z_D) K(x - z_D) f(y) \, d\mathcal{H}^n(y) = 0 \quad \text{for all } 0 < \epsilon \leq \delta.\]

Thus, given \( m \in J \), we can decompose

\[
(K\varphi_{\epsilon_{m+1}} \ast (f\mathcal{H}^n))(x) = \int \varphi_{\epsilon_{m+1}}(x - y) (K(x - y) - K(x - z_D)) f(y) \, d\mathcal{H}^n(y)
\]

\[
+ \int (\varphi_{\epsilon_{m+1}}(x - y) - \varphi_{\epsilon_{m+1}}(x - z_D)) K(x - z_D) f(y) \, d\mathcal{H}^n(y),
\]

and we obtain

\[
|(K\varphi_{\epsilon_{m+1}} \ast (f\mathcal{H}^n))(x)| \leq \|f\|_{L^\infty(\mathcal{H}^n)} (\Theta_1 + \Theta_2),
\]

where

\[
\Theta_1 := \int_D \varphi_{\epsilon_{m+1}}(x - y) \left| K(x - y) - K(x - z_D) \right| \, d\mathcal{H}^n(y),
\]

\[
\Theta_2 := \int_D \left| \varphi_{\epsilon_{m+1}}(x - y) - \varphi_{\epsilon_{m+1}}(x - z_D) \right| \left| K(x - z_D) \right| \, d\mathcal{H}^n(y).
\]

The term \( \Theta_1 \) can be easily handled. For \( x \in \Gamma \setminus 3D \), we have

\[
\Theta_1 \lesssim \ell(D) \text{dist}(x, D)^{n-1} \int_D \varphi_{\epsilon_{m+1}}(x - y) \, d\mathcal{H}^n(y).\]

Let us estimate \( \Theta_2 \). Decompose \( J = S \cup L \), where \( S \) and \( L \) are as in (3.5.4) but replacing \( m \in \mathbb{Z} \) by \( m \in J \), and as before, let \( j(\epsilon_m) \) be the integer such that \( \epsilon_m \in I_{j(\epsilon_m)} \). Using that \( x \in \Gamma \setminus 3D \) and \( \text{supp } f \subset D \), one can easily check that \( L \) contains a finite number of elements, and this number only depends on \( n \) and \( d \). Similarly, \( S_j = \emptyset \) for all \( j \in \mathbb{Z} \) except on a finite number which only depends on \( n \) and \( d \).
Assume that \( m \in \mathcal{S} \). With the same computations as those in (3.4.6), one can prove that, for all \( y \in \text{supp}\tilde{\varphi}^m_{\epsilon+1}(x - \cdot) \), \(|\nabla_y \tilde{\varphi}^m_{\epsilon+1}(x - y)| \lesssim 2^{j(\epsilon_m)} |\epsilon_m - \epsilon_{m+1}| |\bar{x} - \bar{y}|^{-1} \), because \(|\bar{x} - \bar{y}| \approx \epsilon_m \approx \epsilon_{m+1} \approx 2^{-j(\epsilon_m)} \) for all \( y \in \text{supp}\tilde{\varphi}^m_{\epsilon+1}(x - \cdot) \). Thus,

\[
\Theta_{2m} \lesssim \ell(D)^{n+1} \text{dist}(x, D)^{-n} 2^{j(\epsilon_m)} |\epsilon_m - \epsilon_{m+1}|.
\] (3.6.4)

Assume now that \( m \in \mathcal{L} \). It is easy to verify that \(|\nabla_y \tilde{\varphi}^m_{\epsilon+1}(x - y)| \lesssim |\bar{x} - \bar{y}|^{-1} \), so \( \Theta_{2m} \lesssim \ell(D)^{n+1} \text{dist}(x, D)^{-n} \).

Combining this last estimate with (3.6.3), (3.6.4), the fact that \(|(K \tilde{\varphi}^m_{\epsilon+1} * (f \mathcal{H}_\Gamma))(x)| \leq \|f\|_{L^\infty(\mathcal{H}^1_\Gamma)}(\Theta_{1m} + \Theta_{2m})\), the remark on \( \mathcal{S} \) and \( \mathcal{L} \) made just after (3.6.3), (3.6.2), and that \( \rho > 1 \), we have that, for all \( x \in \Gamma \setminus 3D \),

\[
(\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}_\Gamma}_\varphi) f(x) \lesssim \|f\|_{L^\infty(\mathcal{H}^1_\Gamma)} \left( \sum_{m \in \mathcal{L}} \Theta_{1m} + \sum_{m \in \mathcal{S}} \Theta_{2m} + \sum_{m \in \mathcal{L}} \Theta_{2m} \right)
\lesssim \frac{\|f\|_{L^\infty(\mathcal{H}^1_\Gamma)} \ell(D)^{n+1}}{\text{dist}(x, D)^{n+1}} \left( \sum_{m \in \mathcal{L}} \int_D \tilde{\varphi}^m_{\epsilon+1}(x - y) \frac{\ell(D)^n}{\ell(D)^n} d\mathcal{H}^1_\Gamma(y) + \sum_{m \in \mathcal{S}} |\epsilon_m - \epsilon_{m+1}| + \sum_1 \right)
\lesssim \frac{\|f\|_{L^\infty(\mathcal{H}^1_\Gamma)} \ell(D)^{n+1}}{\text{dist}(x, D)^{n+1}}.
\]

Then, using (3.6.1) and standard estimates, we conclude that

\[
\int_{(3D)'\cap \Gamma} (\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}_\Gamma}_\varphi) f(x) \, d\mathcal{H}^1_\Gamma(x) \lesssim \int_{(3D)'\cap \Gamma} \frac{\|f\|_{L^\infty(\mathcal{H}^1_\Gamma)} \ell(D)^{n+1}}{\text{dist}(x, D)^{n+1}} \, d\mathcal{H}^1_\Gamma(x) \lesssim 1.
\]

### 3.6.2 The operator \( \mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}_\Gamma}_\varphi : L^\infty(\mathcal{H}^1_\Gamma) \to BMO(\mathcal{H}^1_\Gamma) \) is bounded

We have to prove that there exists a constant \( C > 0 \) such that, for any \( f \in L^\infty(\mathcal{H}^1_\Gamma) \) and any cube \( \bar{D} \subset \mathbb{R}^n \), there exists some constant \( c \) depending on \( f \) and \( \bar{D} \) such that

\[
\int_{\bar{D}} |(\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}_\Gamma}_\varphi) f - c| \, d\mathcal{H}^n_{\mathcal{H}} \leq C \|f\|_{L^\infty(\mathcal{H}^1_\Gamma)} \mathcal{H}^n_{\mathcal{H}}(D).
\]

Let \( f \) and \( D \) be as above, and set \( f_1 := f \chi_{3D} \) and \( f_2 := f - f_1 \). First of all, by Hölder’s inequality and Theorem 3.5.1, we have

\[
\int_{D} (\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}_\Gamma}_\varphi) f_1 \, d\mathcal{H}^n_{\mathcal{H}} \leq \mathcal{H}^n_{\mathcal{H}}(D)^{1/2} \left( \int_{3D} (\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}_\Gamma}_\varphi) f_1^2 \, d\mathcal{H}^n_{\mathcal{H}} \right)^{1/2}
\lesssim \mathcal{H}^n_{\mathcal{H}}(D)^{1/2} \left( \|f_1\|_{L^\infty(\mathcal{H}^1_\Gamma)} \mathcal{H}^n_{\mathcal{H}}(3D)^{1/2} \right)^{1/2} \lesssim \|f\|_{L^\infty(\mathcal{H}^1_\Gamma)} \mathcal{H}^n_{\mathcal{H}}(D).
\]

Notice that \(|(\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}_\Gamma}_\varphi)(f_1 + f_2) - (\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}_\Gamma}_\varphi)f_2| \leq (\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}_\Gamma}_\varphi)f_1\), because \( \mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}_\Gamma}_\varphi \) is sublinear and positive. Then, for any \( c \in \mathbb{R} \),

\[
|(\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}_\Gamma}_\varphi)(f_1 + f_2) - c| \leq |(\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}_\Gamma}_\varphi)(f_1 + f_2) - (\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}_\Gamma}_\varphi)f_2| + |(\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}_\Gamma}_\varphi)f_2 - c|
\leq (\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}_\Gamma}_\varphi)f_1 + |(\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}_\Gamma}_\varphi)f_2 - c|,
\]

and the desired conclusion follows.
hence we are reduced to prove that, for some constant $c \in \mathbb{R}$,
\[
\int_D \left| (V_\rho \circ T_{\vec{\varphi}}^{H^m_1}) f_2 - c \right| dH^m_1 \leq C \| f \|_{L^\infty(H^m_1)} H^m_1(D).
\] (3.6.5)

Set $z_D := (\tilde{z}_D, A(\tilde{z}_D))$, where $z_D$ is the center of $\tilde{D}$, and take $c := (V_\rho \circ T_{\vec{\varphi}}^{H^m_1}) f_2(z_D)$. We may assume that $c < \infty$ (this is the case if, for example, $f$ has compact support). By the triangle inequality,
\[
\left| (V_\rho \circ T_{\vec{\varphi}}^{H^m_1}) f_2(x) - c \right|^p \leq \sup_{\{\epsilon_m \geq 0\}} \sum_{m \in \mathbb{Z}} \left| (K\varphi_{\epsilon_m+1} * (f_2 H^m_1))(x) - (K\varphi_{\epsilon_m+1} * (f_2 H^m_1))(z_D) \right|^p.
\]

Given $x \in \Gamma \cap D$, let $\{\epsilon_m\}_{m \in \mathbb{Z}}$ be a decreasing sequence of positive numbers (which depends on $x$) such that
\[
\left| (V_\rho \circ T_{\vec{\varphi}}^{H^m_1}) f_2(x) - c \right|^p \leq 2 \sum_{m \in \mathbb{Z}} \left| (K\varphi_{\epsilon_m+1} * (f_2 H^m_1))(x) - (K\varphi_{\epsilon_m+1} * (f_2 H^m_1))(z_D) \right|^p.
\]

Notice that $\left| (K\varphi_{\epsilon_m+1} * (f_2 H^m_1))(x) - (K\varphi_{\epsilon_m+1} * (f_2 H^m_1))(z_D) \right| \leq \| f \|_{L^\infty(H^m_1)} (\Theta_{1m} + \Theta_{2m})$, where $\Theta_{1m}$ and $\Theta_{2m}$ are as in (3.5.10) but replacing $\mu$ by $H^m_1$. It is straightforward to check that the arguments and computations given in Subsection 3.5.2 to estimate the two terms in (3.5.10) (see (3.5.11), (3.5.12), and (3.5.13)) still hold if we replace $\mu$ by $H^m_1$. Therefore, we have
\[
\sum_{m \in \mathbb{Z} = S \cup L} (\Theta_{1m} + \Theta_{2m})^p \lesssim 1,
\]
which implies that $\left| (V_\rho \circ T_{\vec{\varphi}}^{H^m_1}) f_2(x) - c \right| \lesssim \| f \|_{L^\infty(H^m_1)}$ and, by integrating in $D$, gives (3.6.5).

**3.6.3 The operator $V_\rho \circ T_{\vec{\varphi}}^{H^m_1} : L^p(H^m_1) \rightarrow L^p(H^m_1)$ is bounded for all $1 < p < \infty$**

Since $V_\rho \circ T_{\vec{\varphi}}^{H^m_1}$ is sublinear, the $L^p$ boundedness follows by applying the results of Subsections 3.6.1 and 3.6.2, and the interpolation theorem in [Ju, page 43] between the pairs $(H^1(H^m_1), L^1(H^m_1))$ and $(L^\infty(H^m_1), BMO(H^m_1))$.

Given a $v$-cube $Q \subset \mathbb{R}^d$, set $m_Q(f) := H^m_1(Q)^{-1} \int_Q f \, dH^m_1$, and let $M$ denote the Hardy-Littlewood maximal operator with respect to $\Gamma$, i.e. for $x \in \Gamma$, $M(f)(x) := \sup m_Q(|f|)$, where the supremum is taken over all $v$-cubes $Q \subset \mathbb{R}^d$ containing $x \in \Gamma$. Let $M^2$ be the sharp maximal operator defined by $M^2(f)(x) := \sup m_Q(|f - m_Q(f)|)$, where the supremum is also taken over all $v$-cubes $Q \subset \mathbb{R}^d$ containing $x \in \Gamma$.

One comment about the interpolation theorem in [Ju, page 43] is in order. Given an operator $F$ bounded form $H^1$ to $L^1$ and from $L^\infty$ to $BMO$, in the proof of the interpolation theorem applied to $F$, one uses that $M^2 \circ F$ is sublinear (i.e. $(M^2 \circ F)(f + g) \leq (M^2 \circ F)f +$
3.6. \( L^p \) and endpoint estimates for \( \mathcal{V}_\rho \circ \mathcal{T}_\varphi^{H^p} \) and \( \mathcal{O} \circ \mathcal{T}_\varphi^{H^p} \)

(M^\sharp \circ F)g for all functions \( f, g \). This is the case when \( F \) is linear. However, \( \mathcal{V}_\rho \circ \mathcal{T}_\varphi^{H^p} \) is not linear, and then it is not clear if \( M^\sharp \circ \mathcal{V}_\rho \circ \mathcal{T}_\varphi^{H^p} \) is sublinear. Nevertheless, this problem can be fixed easily using that \( \mathcal{V}_\rho \circ \mathcal{T}_\varphi^{H^p} \) is sublinear and positive (that is \( (\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{H^p})f(x) \geq 0 \) for all \( f \) and \( x \)), as the following lemma shows.

**Lemma 3.6.2.** Let \( F : L^1_{loc}(H^p_\Gamma) \to L^1_{loc}(H^p_\Gamma) \) be a positive and sublinear operator. Then

\[
(M^\sharp \circ F)(f + g) \lesssim (M \circ F)f + (M^\sharp \circ F)g
\]

for all functions \( f, g \).

**Proof.** If \( F \) is sublinear and positive, one has that \( |F(f)(x) - F(g)(x)| \leq F(f - g)(x) \) for all functions \( f, g \in L^1_{loc}(H^p_\Gamma) \). Then, for \( x, y \in Q \cap \Gamma \),

\[
|F(f + g)(y) - m_Q(Fg)| \leq |F(f + g)(y) - Fg(y)| + |Fg(y) - m_Q(Fg)| \\
\leq |Ff(y)| + |Fg(y) - m_Q(Fg)|.
\]

Hence, \( m_Q|F(f + g) - m_Q(Fg)| \leq m_Q|Ff| + m_Q|Fg - m_Q(Fg)| \leq (M \circ F)f(x) + (M^\sharp \circ F)g(x) \) and, by taking the supremum over all possible v-cubes \( Q \ni x \), we conclude \( (M^\sharp \circ F)(f + g)(x) \lesssim (M \circ F)f(x) + (M^\sharp \circ F)g(x) \). \( \square \)

By using Lemma 3.6.2 and the fact that \( \|Mf\|_{L^p(H^p_\Gamma)} \lesssim \|M^\sharp f\|_{L^p(H^p_\Gamma)} \) for \( f \in L^{p_0}(H^p_\Gamma) \cap L^p(H^p_\Gamma) \) and \( 1 \leq p_0 \leq p < \infty \), one can reprove Journé’s interpolation theorem applied to \( \mathcal{V}_\rho \circ \mathcal{T}_\varphi^{H^p} \) with minor modifications in the original proof.
Chapter 4

Variation and oscillation for singular integrals with odd kernel on Lipschitz graphs: Rough truncation

We continue with the proof of Main Theorem 3.0.1. In Chapter 3, we proved Main Theorem 3.0.1 for the case $\omega = \tilde{\varphi}$. The purpose of this chapter is to prove Main Theorem 3.0.1 for the other cases $\omega \in \{\chi, \tilde{\chi}, \varphi\}$. As we said in the introduction, the reader mostly interested in the principal result of Chapter 5 can skip this chapter (see also Remark 5.0.14).

4.1 $L^2$ estimates for $\mathcal{V}_\rho \circ \mathcal{T}_\chi^{\mathcal{H}_x^n}$ and $\mathcal{O} \circ \mathcal{T}_\chi^{\mathcal{H}_x^n}$

Recall that $\Gamma := \{x \in \mathbb{R}^d : x = (\tilde{x}, A(\tilde{x}))\}$ and we denote the Lipschitz constant of $A$ by Lip($A$). Let $A_\Gamma : \mathbb{R}^n \to \mathbb{R}^d$ be the parametrization of $\Gamma$, i.e. $A_\Gamma(y) := (y, A(y))$ for $y \in \mathbb{R}^n$. We may and will assume that the Lipschitz function $A$ has compact support, and our estimates will not depend on the support of $A$. By a limiting argument, one easily obtains the same estimates for the case of a general Lipschitz graph.

Abusing notation, throughout this section we will identify the cubes $D \subset \mathbb{R}^n$ with the $v$-cubes $D \times \mathbb{R}^{d-n} \subset \mathbb{R}^d$, so we will use the same symbol $D$ to denote both objects. In particular, $\mathcal{D}$ will denote the dyadic lattice of cubes in $\mathbb{R}^n$ and the dyadic lattice of $v$-cubes in $\mathbb{R}^d$. It will be clear from the context to which object we are referring to in each particular circumstance. Recall that we have set $\| \cdot \|_p := \| \cdot \|_{L^p(\mathcal{L}^n)}$ for $1 \leq p \leq \infty$, and $dy := d\mathcal{L}^n(y)$ for $y \in \mathbb{R}^n$.

The aim of this section is the following theorem:

**Theorem 4.1.1.** Let $\rho > 2$, and assume Lip($A) < 1$. The operators $\mathcal{V}_\rho \circ \mathcal{T}_\chi^{\mathcal{H}_x^n}$ and $\mathcal{O} \circ \mathcal{T}_\chi^{\mathcal{H}_x^n}$ are bounded in $L^2(\mathcal{H}_x^n)$, and the norm of $\mathcal{O} \circ \mathcal{T}_\chi^{\mathcal{H}_x^n}$ is bounded above independently of the
4. Variation for singular integrals on Lipschitz graphs: Rough truncation

sequence that defines $\mathcal{O}$.

As in the previous sections, we will only prove Theorem 4.1.1 in the case of the $\rho$-variation, because the proof for the oscillation follows by very similar arguments. Moreover, we will use the following lemma, which is proved in Subsection 4.1.6 (see Lemma 4.1.9):

**Lemma 4.1.2.** If $\text{Lip}(A) < 1$, there exists $C > 0$ depending on $\text{Lip}(A)$ such that

$$\mathcal{H}_1^n(A^n(d(z, a, b)) \leq C(b - a)b^{n-1} \quad \text{for all } 0 < a \leq b \text{ and } z \in \Gamma.$$  \hfill (4.1.1)

**Remark 4.1.3.** Without the assumption $\text{Lip}(A) < 1$, the lemma fails (see Remark 4.1.11 below). The estimate (4.1.1) is essential for some of the arguments below. But, as we will see in Chapter 5, it is only a technical obstruction in the proof of Theorem 4.1.1 that can be avoided by using more sophisticated techniques (like the so-called corona decomposition, for example).

This lemma is only required to study the $\rho$-variation and oscillation for singular integrals when the family of truncations is $\chi$. If we considered the family $\tilde{\chi}$ instead of $\chi$, we would have to estimate $\mathcal{H}_1^n(A^n(z, a, b) \times \mathbb{R}^{d-n})$, which is easily seen to be bounded by $C(b - a)b^{n-1}$ in any case, i.e. without the extra assumption $\text{Lip}(A) < 1$. On the other hand, if we worked with $\varphi$, we would not need to estimate the size of any annulus in our computations. Instead, we would use the regularity of the functions $\varphi_\epsilon$ for $\epsilon > 0$, as we did with $\tilde{\varphi}_\epsilon$ in the preceding sections. For more details, see Remark 4.1.7.

### 4.1.1 Beginning of the proof of Theorem 4.1.1

Let $f \in L^2(\mathcal{H}^n_1)$. Given $x \in \Gamma$, let $\{\epsilon_m\}_{m \in \mathbb{Z}}$ be a decreasing sequence of positive numbers (which depends on $x$) such that

$$((\mathcal{V}_\rho \circ T_{\chi}^{\mathcal{H}^n_1})f(x))^\rho \leq 2 \sum_{m \in \mathbb{Z}} |(K\chi_{\epsilon_m} \ast (f\mathcal{H}^n_1))(x)|^\rho.$$ \hfill (4.1.2)

Given $j \in \mathbb{Z}$, we denote $I_j := [2^{-j-1}, 2^{-j})$. Let $j(\epsilon_m)$ be the integer such that $\epsilon_m \in I_{j(\epsilon_m)}$. As before, we set $S := \bigcup_{j \in \mathbb{Z}} S_j, S_j := \{m \in \mathbb{Z} : \epsilon_m, \epsilon_{m+1} \in I_j\}$, and $\mathcal{L} := \{m \in \mathbb{Z} : \epsilon_m \in I_i, \epsilon_{m+1} \in I_j \text{ for } i < j \}.$

For $\epsilon > 0$, we define $\kappa_\epsilon := \chi_\epsilon - \tilde{\varphi}_\epsilon$. Then, by (4.1.2) and the triangle inequality,

$$((\mathcal{V}_\rho \circ T_{\chi}^{\mathcal{H}^n_1})f(x))^\rho \leq \sum_{m \in S} |(K\chi_{\epsilon_m} \ast (f\mathcal{H}^n_1))(x)|^\rho + \sum_{m \in \mathcal{L}} |(K\kappa_{\epsilon_m} \ast (f\mathcal{H}^n_1))(x)|^\rho + \sum_{m \in \mathcal{L}} |(K\tilde{\varphi}_{\epsilon_m} \ast (f\mathcal{H}^n_1))(x)|^\rho.$$ \hfill (4.1.3)

Notice that $\sum_{m \in \mathcal{L}} |(K\tilde{\varphi}_{\epsilon_m} \ast (f\mathcal{H}^n_1))(x)|^\rho \leq ((\mathcal{V}_\rho \circ T_{\tilde{\varphi}}^{\mathcal{H}^n_1})f(x))^\rho.$
We will prove the following estimate in Subsections 4.1.3, 4.1.4 and 4.1.5:

\[
\sum_{j \in \mathbb{Z}} \sum_{D \in \mathcal{D}_j} \int_D \sum_{m \in S_j} |(K \chi_{\epsilon m} \ast (f \mathcal{H}^n_T))|^2 \, d\mathcal{H}^n_T
\]

\[
+ \sum_{j \in \mathbb{Z}} \sum_{D \in \mathcal{D}_j} \int_D \sum_{m \in \mathcal{L} : \epsilon m \in I_j} |(K \kappa_{\epsilon m} \ast (f \mathcal{H}^n_T))|^2 \, d\mathcal{H}^n_T
\]

\[
+ \sum_{j \in \mathbb{Z}} \sum_{D \in \mathcal{D}_j} \int_D \sum_{m \in \mathcal{L} : \epsilon m+1 \in I_j} |(K \kappa_{\epsilon m+1} \ast (f \mathcal{H}^n_T))|^2 \, d\mathcal{H}^n_T \lesssim \|f\|^2_{L^2(\mathcal{H}^n_T)}.
\]

(4.1.4)

Using (4.1.4), (4.1.3), Theorem 3.6.1 for \( p = 2 \), and that \( \rho > 2 \), we finally get

\[
\|(V_\rho \circ \mathcal{T}^{H^\rho}_T f\|_{L^2(\mathcal{H}^n_T)} \lesssim \|(V_\rho \circ \mathcal{T}^{H^\rho}_T f\|_{L^2(\mathcal{H}^n_T)} + \int \left( \sum_{m \in \mathcal{S}} |(K \chi_{\epsilon m} \ast (f \mathcal{H}^n_T))|^\rho \right)^{2/\rho} \, d\mathcal{H}^n_T
\]

\[
+ \int \left( \sum_{m \in \mathcal{L}} |(K \kappa_{\epsilon m} \ast (f \mathcal{H}^n_T))|^\rho \right)^{2/\rho} \, d\mathcal{H}^n_T + \int \left( \sum_{m \in \mathcal{L}} |(K \kappa_{\epsilon m+1} \ast (f \mathcal{H}^n_T))|^\rho \right)^{2/\rho} \, d\mathcal{H}^n_T
\]

\[
\lesssim \|f\|^2_{L^2(\mathcal{H}^n_T)} + \sum_{j \in \mathbb{Z}} \sum_{D \in \mathcal{D}_j} \int_D \sum_{m \in S_j} |(K \chi_{\epsilon m} \ast (f \mathcal{H}^n_T))|^2 \, d\mathcal{H}^n_T
\]

\[
+ \sum_{j \in \mathbb{Z}} \sum_{D \in \mathcal{D}_j} \int_D \sum_{m \in \mathcal{L} : \epsilon m \in I_j} |(K \kappa_{\epsilon m} \ast (f \mathcal{H}^n_T))|^2 \, d\mathcal{H}^n_T
\]

\[
+ \sum_{j \in \mathbb{Z}} \sum_{D \in \mathcal{D}_j} \int_D \sum_{m \in \mathcal{L} : \epsilon m+1 \in I_j} |(K \kappa_{\epsilon m+1} \ast (f \mathcal{H}^n_T))|^2 \, d\mathcal{H}^n_T \lesssim \|f\|^2_{L^2(\mathcal{H}^n_T)}.
\]

It only remains to prove (4.1.4).

### 4.1.2 Estimate of \( \sum_{m \in S_j} |(K \chi_{\epsilon m} \ast (f \mathcal{H}^n_T))(x)|^2 \) for \( x \in \Gamma \cap D \) and \( D \in \mathcal{D}_j \)

Using the parametrization \( A_\Gamma \) of \( \Gamma \), we have

\[
(K \chi_{\epsilon m} \ast (f \mathcal{H}^n_T))(x) = \int_{\mathbb{R}^n} K(x - A_\Gamma(y)) \chi_{\epsilon m+1}(x - A_\Gamma(y)) f(A_\Gamma(y)) |J(A_\Gamma)(y)| \, dy
\]

\[
= \int_{\mathbb{R}^n} K(x - A_\Gamma(y)) \chi_{\epsilon m+1}(x - A_\Gamma(y)) g(y) \, dy,
\]

where \( J(A_\Gamma) \) stands for the \( n \)-dimensional Jacobian of the map \( A_\Gamma := y \mapsto (y, A(y)) \), and we have set \( g(y) := f(A_\Gamma(y)) |J(A_\Gamma)(y)| \) for \( y \in \mathbb{R}^n \) (notice that \( x \in \mathbb{R}^d \) but \( y \in \mathbb{R}^n \), we have not used the notation \( \tilde{y} \) here to make it simpler). Since \( \Gamma \) is a Lipschitz graph \( |J(A_\Gamma)| \approx 1 \), so \( g \in L^2(\mathcal{L}^n) \).

**Definition 4.1.4.** Let \( \{\psi_Q^k\}_{Q \in \mathcal{D}, k = 1, \ldots, 2^n - 1} \) be an orthonormal basis of \( \mathcal{C}^1 \) wavelets on \( \mathbb{R}^n \) in the following manner (see [Da1, Part I]):

(a) \( \psi_Q^k : \mathbb{R}^n \to \mathbb{R} \) is a \( \mathcal{C}^1 \) function for all \( Q \in \mathcal{D} \) and \( k = 1, \ldots, 2^n - 1 \).
(b) There exists $C > 1$ and $\psi_0 : [0, C]^n \to \mathbb{R}$ with $\|\psi_0\|_2 = 1$, $\|\psi_0\|_\infty \lesssim 1$, and such that, for any $Q \in \mathcal{D}$ and $k = 1, \ldots, 2^n - 1$, there exists $l \in \mathbb{Z}^n$ such that $\psi^k_Q(y) = \psi_0(y/\ell(Q) - l)\ell(Q)^{-n/2}$ for all $y \in \mathbb{R}^n$.

(c) $\|\psi^k_Q\|_2 = 1$, $\int \psi^k_Q d\mathcal{L}^n = 0$ and $\int \psi^k_Q \psi^l_R d\mathcal{L}^n = 0$, for all $Q, R \in \mathcal{D}$ and $k, l = 1, \ldots, 2^n - 1$ such that $(Q, k) \neq (R, l)$.

(d) $\text{supp}\psi^k_Q \subset \text{supp}\psi^k_Q$ for all $Q \in \mathcal{D}$ and $k = 1, \ldots, 2^n - 1$, where $C_w > 1$ is some fixed constant (which depends on $n$).

(e) $\|\psi^k_Q\|_\infty \lesssim \ell(Q)^{-n/2}$ and $\|\nabla \psi^k_Q\|_\infty \lesssim \ell(Q)^{-n/2-1}$ for all $Q \in \mathcal{D}$, $k = 1, \ldots, 2^n - 1$.

(f) If $h \in L^2(\mathcal{L}^n)$, then $h = \sum_{Q \in \mathcal{D}, k=1,\ldots,2^n-1} \Delta^k_Q h$, where $\Delta^k_Q h := (\int h \psi^k_Q d\mathcal{L}^n) \psi^k_Q$.

In order to reduce the notation, we may think that a cube of $\mathcal{D}$ is not only a subset of $\mathbb{R}^n$, but a couple $(Q, k)$, where $Q$ is a subset of $\mathbb{R}^n$ and $k = 1, \ldots, 2^n - 1$. In particular, there exist $2^n - 1$ cubes in $\mathcal{D}$ such that the subsets that they represent in $\mathbb{R}^n$ coincide. We make this abuse of notation to avoid using the superscript $k$ in the previous definition. Then, we can rewrite the wavelet basis as $\{\psi_Q\}_{Q \in \mathcal{D}}$, with the evident adjustments of the properties $(a), \ldots, (f)$ in Definition 4.1.4.

**Remark 4.1.5.** Since $\Gamma$ is Lipschitz graph, $|J(A_\Gamma)(y)| \approx 1$ for all $y \in \mathbb{R}^n$. Then, using Definition 4.1.4(c) and Definition 4.1.4(f), one easily obtains $\|f\|_{L^2(\mathcal{H}_\Gamma)}^2 \approx \|g\|^2 = \sum_{Q \in \mathcal{D}} \|\Delta_Q g\|^2$.

Given $x \in \mathcal{D} \cap \Gamma$, if $m \in \mathcal{S}$ and $\text{supp}\psi_Q \cap \text{supp} \chi^{t_m}_{t_{m+1}}(x - A_\Gamma(\cdot)) \neq \emptyset$, then either $D \subset C_b Q$ or $Q \subset C_b D$, where $C_b$ is some big fixed constant. Set $J := \{Q \in \mathcal{D} : D \subset C_b Q \text{ and } Q \not\subset C_b D\}$ and $\Psi_{DQ} := \sum_{Q \in J} \Delta_Q g$. Using (4.1.5) and Definition 4.1.4(f), we have

\[
(K \chi^{t_m}_{t_{m+1}} * (f \mathcal{H}_\Gamma^n))(x) = \int_{\mathbb{R}^n} K(x - A_\Gamma(y)) \chi^{t_m}_{t_{m+1}}(x - A_\Gamma(y)) g(y) dy \\
= \int_{\mathbb{R}^n} K(x - A_\Gamma(y)) \chi^{t_m}_{t_{m+1}}(x - A_\Gamma(y)) (\Psi_{DQ} g - \Psi_D g(x)) dy \\
+ \Psi_D g(x) \int_{\mathbb{R}^n} K(x - A_\Gamma(y)) \chi^{t_m}_{t_{m+1}}(x - A_\Gamma(y)) dy \\
+ \sum_{Q \in \mathcal{D}} \int_{\mathbb{R}^n} K(x - A_\Gamma(y)) \chi^{t_m}_{t_{m+1}}(x - A_\Gamma(y)) \Delta_Q g(y) dy \\
=: U_{1m}(x) + U_{2m}(x) + U_{3m}(x).
\]
4.1.2.1 Estimate of $\sum_{m \in S_j} |U_1^m(x)|^2$

Notice that, by Definition 4.1.4(c), $\|\nabla (\Delta Qg)\|_\infty \lesssim \|\Delta Qg\|_2 \ell(Q)^{-n/2-1}$. Then,

$$|U_1^m(x)| \leq \sum_{Q \in J} \int_{\mathbb{R}^n} |K(x - A \Gamma(y))| \chi_{m+1}^e(x - A \Gamma(y))|\Delta Qg(y) - \Delta Qg(x)| \, dy$$

$$\leq \sum_{Q \in J} \int_{\mathbb{R}^n} |K(x - A \Gamma(y))| \chi_{m+1}^e(x - A \Gamma(y))|\nabla (\Delta Qg)\|_\infty |x - y| \, dy$$

$$\lesssim \sum_{Q \in J} \ell(D)^{-n+1} \|\Delta Qg\|_2 \ell(Q)^{-n/2-1} \int_{\mathbb{R}^n} \chi_{m+1}^e(x - A \Gamma(y)) \, dy,$$

thus, by Cauchy-Schwarz inequality and the fact that $\sum_{m \in S_j} \int_{\mathbb{R}^n} \chi_{m+1}^e(x - A \Gamma(y)) \, dy \lesssim \ell(D)^n$,

$$\sum_{m \in S_j} |U_1^m(x)|^2 \lesssim \left( \sum_{Q \in D_m : D_m \subset C_m Q} \ell(Q)^{-n/2} \ell(D) \|\Delta Qg\|_2 \right)^2 \lesssim \sum_{Q \in D_m : D_m \subset C_m Q} \frac{\ell(D)}{\ell(Q)^{n+1}} \|\Delta Qg\|_2^2. \quad (4.1.7)$$

4.1.2.2 Estimate of $\sum_{m \in S_j} |U_3^m(x)|^2$

To estimate $U_3^m(x)$, we denote

$$U_3^m(x, Q) := \int \chi_{m+1}^e(x - A \Gamma(y)) K(x - A \Gamma(y)) \Delta Qg(y) \, dy,$$

so $U_3^m(x) = \sum_{Q \in D : Q \subset C_m D} U_3^m(x, Q)$. Notice that, if $C_m Q \cap \Gamma \cap A(x, \epsilon_{m+1}, \epsilon_m) = \emptyset$, then $U_3^m(x, Q) = 0$. So, if we set

$$J_m^1 := \{ Q \in D : Q \subset C_m D, C_m Q \cap \Gamma \cap A(x, \epsilon_{m+1}, \epsilon_m) \neq \emptyset, 100 \sqrt{n} C_m \ell(Q) \geq \epsilon_m - \epsilon_{m+1} \},$$

$$J_m^2 := \{ Q \in D : Q \subset C_m D, C_m Q \cap \Gamma \cap A(x, \epsilon_{m+1}, \epsilon_m) \neq \emptyset, 100 \sqrt{n} C_m \ell(Q) < \epsilon_m - \epsilon_{m+1} \},$$

we have

$$U_3^m(x) = \sum_{Q \in J_m^1} U_3^m(x, Q) + \sum_{Q \in J_m^2} U_3^m(x, Q). \quad (4.1.8)$$

If $m \in S_j$ and $Q \in J_m^1$, we have $\int \chi_{m+1}^e(x - A \Gamma(y))|\Delta Qg(y)| \, dy \lesssim |\epsilon_m - \epsilon_{m+1}| \|\Delta Qg\|_1/\ell(Q)$. This follows from the smoothness of the wavelet $\psi_Q$, Definition 4.1.4(b), and Lemma 4.1.2. Then,

$$|U_3^m(x, Q)| \lesssim \int \chi_{m+1}^e(x - A \Gamma(y)) \ell(D)^{-n} |\Delta Qg(y)| \, dy \lesssim \frac{|\epsilon_m - \epsilon_{m+1}|}{\ell(D)^n} \|\Delta Qg\|_1$$

$$\lesssim \frac{\ell(Q)^{n/2-1}}{\ell(D)^n} |\epsilon_m - \epsilon_{m+1}| \|\Delta Qg\|_2.$$
Therefore, by Cauchy-Schwarz,

\[
\sum_{m \in \mathcal{S}_j} \left( \sum_{Q \in J^1_m} |U^3_m(x, Q)| \right)^2 \lesssim \sum_{m \in \mathcal{S}_j} \left( \sum_{Q \in J^1_m} \frac{\ell(Q)^{n-1}}{\ell(D)^n} |\epsilon_m - \epsilon_{m+1}| \| \Delta Qg \|_2 \right)^2 \leq \frac{1}{\ell(D)^{2n}} \sum_{m \in \mathcal{S}_j} \left( \sum_{Q \in J^1_m} \ell(Q)^{n-1} \right) \left( \sum_{Q \in J^1_m} \frac{|\epsilon_m - \epsilon_{m+1}|^2}{\ell(Q)} \| \Delta Qg \|_2^2 \right). 
\]

From the definition of \( J^1_m \), it is not difficult to check that

\[
\sum_{Q \in J^1_m} \ell(Q)^{n-1} \lesssim \ell(D)^{n-1} \log_2 (\ell(D)/|\epsilon_m - \epsilon_{m+1}|). 
\]

To check this, recall that \( \epsilon_m, \epsilon_{m+1} \in I_j, D \in D_j, \) and \( Q \in J^1_m \). Then, split the sum according to the different scales of the v-cubes and use that, given \( i \in \mathbb{Z} \) such that \( \sqrt{\pi} C_w 2^{-1} \geq |\epsilon_m - \epsilon_{m+1}| \), the number of v-cubes \( Q \in \mathcal{D} \) such that \( \ell(Q) = 2^{-i}, Q \subset C_b D, \) and \( C_w Q \cap \Gamma \cap A(x, \epsilon_{m+1}, \epsilon_m) \neq \emptyset \) is bounded by \( C \ell(D)^{n-1} 2^{-n-i} \), since for all these v-cubes, \( C_w Q \cap \Gamma \subset A(x, \epsilon_{m+1} - C2^{-i}, \epsilon_m + C2^{-i}) \cap \Gamma \), and by Lemma 4.1.2, \( H^0(\{A(x, \epsilon_{m+1} - C2^{-i}, \epsilon_m + C2^{-i}) \}) \lesssim \ell(D)^{n-1} 2^{-i} \).

Then, by (4.1.9) and since \( t^{1/2} \log_2 (1/t) \lesssim 1 \) for all \( 0 < t \lesssim 1 \),

\[
\sum_{m \in \mathcal{S}_j} \left( \sum_{Q \in J^1_m} |U^3_m(x, Q)| \right)^2 \lesssim \frac{1}{\ell(D)^{n+1}} \sum_{m \in \mathcal{S}_j} \log_2 \left( \frac{\ell(D)}{|\epsilon_m - \epsilon_{m+1}|} \right) \sum_{Q \in J^1_m} \frac{|\epsilon_m - \epsilon_{m+1}|^2}{\ell(Q)} \| \Delta Qg \|_2^2 \lesssim \frac{1}{\ell(D)^n} \sum_{m \in \mathcal{S}_j} \sum_{Q \in J^1_m} \frac{|\epsilon_m - \epsilon_{m+1}|^{3/2}}{\ell(Q)^{1/2} \ell(D)^{n-2}} \| \Delta Qg \|_2^2 \lesssim \sum_{Q \in \mathcal{D}: Q \subset C_b D, m \in \mathcal{S}_j: 100 \sqrt{\pi} C_w \ell(Q) \geq |\epsilon_m - \epsilon_{m+1}|} \frac{\ell(Q)^{1/2}}{\ell(D)^{n+1/2}} \| \Delta Qg \|_2^2.
\]

Since \( \sum_{m \in \mathcal{S}_j: 100 \sqrt{\pi} C_w \ell(Q) \geq |\epsilon_m - \epsilon_{m+1}|} |\epsilon_m - \epsilon_{m+1}|/\ell(Q) \lesssim 1 \), we finally obtain

\[
\sum_{m \in \mathcal{S}_j} \left( \sum_{Q \in J^1_m} |U^3_m(x, Q)| \right)^2 \lesssim \sum_{Q \in \mathcal{D}: Q \subset C_b D} \frac{\ell(Q)^{1/2}}{\ell(D)^{n+1/2}} \| \Delta Qg \|_2^2. 
\]

Assume now that \( Q \in J^2_m \). Let \( z_q \) be the center of \( Q \subset \mathbb{R}^n \). Since \( \int \Delta Qg(y) \, dy = 0 \) (see Definition 4.1.4(c)), we decompose

\[
U^3_m(x, Q) = \int \chi_{\epsilon_{m+1}}(x - A_T(y))(K(x - A_T(y)) - K(x - A_T(z_q))) \Delta Qg(y) \, dy
+ \int \chi_{\epsilon_{m+1}}(x - A_T(y)) - \chi_{\epsilon_{m+1}}(x - A_T(z_q)) K(x - A_T(z_q)) \Delta Qg(y) \, dy
=: U^3_{A,m}(x, Q) + U^3_{B,m}(x, Q).
\]
The first term of the previous sum can be easily handled:

\[
\sum_{m \in S_j} \left( \sum_{Q \in J_m} |U^3_{m}(x,Q)| \right)^2 \lesssim \left( \sum_{m \in S_j} \sum_{Q \in J_m} \frac{\ell(Q)}{\ell(D)^{n+1}} \int \chi_{i_{m+1}}(x-A(x,y))|\Delta_Q g(y)| \, dy \right)^2 \\
\leq \left( \sum_{Q \in D : Q \subset D} \frac{\ell(Q)}{\ell(D)^{n+1}} \|\Delta_Q g\|_1 \right)^2 \lesssim \left( \sum_{Q \in D : Q \subset D} \frac{\ell(Q)^{n+1}}{\ell(D)^{n+1}} \|\Delta_Q g\|_2 \right)^2.
\]

Then, using Cauchy-Schwarz and that \( \sum_{Q \in D : Q \subset D} \ell(Q)^{n+1} \lesssim \ell(D)^{n+1} \), we conclude

\[
\sum_{m \in S_j} \left( \sum_{Q \in J_m} |U^3_{m}(x,Q)| \right)^2 \lesssim \left( \sum_{Q \in D : Q \subset D} \frac{\ell(Q)^{n+1}}{\ell(D)^{2n+2}} \|\Delta_Q g\|_2 \right)^2 \\
\lesssim \sum_{Q \in D : Q \subset D} \frac{\ell(Q)}{\ell(D)^{n+1}} \|\Delta_Q g\|_2^2 \lesssim \sum_{Q \in D : Q \subset D} \frac{\ell(Q)^{1/2}}{\ell(D)^{n+1/2}} \|\Delta_Q g\|_2^2.
\]

(4.1.13)

To deal with \( U^3_{m}(x,Q) \), notice that, if \( C_{w}Q \cap \Gamma \cap \partial A(x, \epsilon_{m+1}, \epsilon_m) = \emptyset \), then \( \chi_{\epsilon_{m+1}}(x-A(x,y)) - \chi_{\epsilon_m}(x-A(x,z)) = 0 \) for all \( y \in C_{w}Q \subset \mathbb{R}^n \). So, \( \sum_{Q \in J_m} |U^3_{m}(x,Q)| = \sum_{Q \in J_m} |U^3_{m}(x,Q)| \), where

\[ J_m^3 := \{ Q \in D : Q \subset C_{b}D, C_{w}Q \cap \Gamma \cap \partial A(x, \epsilon_{m+1}, \epsilon_m) \neq \emptyset, 100\sqrt{n}C_w\ell(Q) < \epsilon_m - \epsilon_{m+1} \}. \]

We use the easy estimate \( |U^3_{m}(x,Q)| \lesssim \ell(D)^{-n} \|\Delta_Q g\|_1 \lesssim \ell(Q)^{n/2} \ell(D)^{-n} \|\Delta_Q g\|_2 \) for all \( Q \in J_m^3 \) and then, by Cauchy-Schwarz inequality,

\[
\left( \sum_{Q \in J_m^3} |U^3_{m}(x,Q)| \right)^2 \lesssim \ell(D)^{-2n} \left( \sum_{Q \in J_m^3} \ell(Q)^{n-1/2} \right) \left( \sum_{Q \in J_m^3} \ell(Q)^{1/2} \|\Delta_Q g\|_2 \right)^2.
\]

(4.1.14)

It is not difficult to show that \( \sum_{Q \in J_m^3} \ell(Q)^{n-1/2} \lesssim \ell(D)^{n-1} |\epsilon_m - \epsilon_{m+1}|^{1/2} \). To check this, split the sum according to the different scales of the v-cubes and use that, given \( i \in \mathbb{Z} \) such that \( 100\sqrt{n}C_w2^{-i} < \epsilon_m - \epsilon_{m+1} \), the number of v-cubes \( Q \subset D \) such that \( \ell(Q) = 2^{-i} \) and \( C_{w}Q \cap \Gamma \cap \partial B(x, \epsilon_{m+1}) \neq \emptyset \) or \( C_{w}Q \cap \Gamma \cap \partial B(x, \epsilon_m) \neq \emptyset \) is bounded by \( C\ell(D)^{(n-1)2^{i(n-1)}} \) due to Lemma 4.1.2, arguing as below (4.1.10). Further, for a fixed \( Q \subset D \) such that \( Q \subset C_{b}D \),

\[
\sum_{m \in S_j : 100\sqrt{n}C_w\ell(Q) < \epsilon_m - \epsilon_{m+1}, C_{w}Q \cap \Gamma \cap \partial A(x, \epsilon_{m+1}, \epsilon_m) \neq \emptyset} \frac{|\epsilon_m - \epsilon_{m+1}|^{1/2}}{\ell(D)^{1/2}} \lesssim 1,
\]

because \( |\epsilon_m - \epsilon_{m+1}| < \ell(D) \) and the sum contains finitely many terms, depending only on \( d \), \( n \) and Lip(A). Applying these remarks in (4.1.14) and interchanging the order of summation,
we obtain

\[ \sum_{m \in S_j} \left( \sum_{Q \in J_n^m} |U^{3B}_m(x, Q)| \right)^2 \lesssim \sum_{m \in S_j} \ell(D)^{-n-1} |\epsilon_m - \epsilon_{m+1}|^{1/2} \sum_{Q \in J_n^m} \ell(Q)^{1/2} \|\Delta_\mu g\|_2^2 \]

\[ = \sum_{m \in S_j} \sum_{Q \in J_n^m} \frac{|\epsilon_m - \epsilon_{m+1}|^{1/2}}{\ell(D)^{1/2}} \frac{\ell(Q)^{1/2}}{\ell(D)^{n+1/2}} \|\Delta_\mu g\|_2^2 \lesssim \sum_{Q \in D : Q \subset C_\ell D} \frac{\ell(Q)^{1/2}}{\ell(D)^{n+1/2}} \|\Delta_\mu g\|_2^2. \]  

(4.1.15)

Finally, combining (4.1.8), (4.1.11), (4.1.12), (4.1.13), and (4.1.15), we conclude

\[ \sum_{m \in S_j} |U^{3m}_m(x)|^2 \lesssim \sum_{Q \in D : Q \subset C_\ell D} \frac{\ell(Q)^{1/2}}{\ell(D)^{n+1/2}} \|\Delta_\mu g\|_2^2. \]  

(4.1.16)

4.1.2.3 Estimate of \( \sum_{m \in S_j} |U^{2m}_m(x)|^2 \)

Let \( L_D \) be a minimizing \( n \)-plane for \( \alpha_x(D) \) and let \( L^x_D \) be the \( n \)-plane parallel to \( L_D \) which contains \( x \). Let \( A_x : \mathbb{R}^n \to L^x_D \) be the parametrization of the \( n \)-plane \( L^x_D \) given by \( A_x(z) := (z, a_x(z)) \) where \( a_x : \mathbb{R}^n \to \mathbb{R}^{d-n} \) is some affine map, and let \( |J(A_x)| \) denote its \( n \)-dimensional Jacobian (notice that \( |J(A_x)| \) is bounded by some constant depending only on \( \text{Lip}(A) \)). Given \( z \in \mathbb{R}^d \), let \( p_0^z \) denote the orthogonal projection onto \( L^x_D \), and for \( z \in \mathbb{R}^d \setminus (p_0^z)^{-1}(x) \), consider the angular projection given by

\[ p^x(z) := x + (p_0^z(z) - x) \frac{|z - x|}{|p_0^z(z) - x|}. \]  

(4.1.17)

If \( z \in \mathbb{R}^d \setminus (p_0^z)^{-1}(x) \), then \( |p_0^z(z) - x| \neq 0 \), so \( p^x \) is well defined and \( |z - x| = |p^x(z) - x| \).

Since \( \Gamma \) is a Lipschitz graph with slope strictly less than 1, we have \( (p_0^z)^{-1}(x) \cap \Gamma = \{x\} \), because the slope of the \( n \)-plane \( L^x_D \) is also smaller than 1 and then the \((d-n)\)-plane passing through \( x \) and orthogonal to \( L^x_D \) does not intersect the cone

\[ \{ y \in \mathbb{R}^d : |(y - \bar{y}) - (x - \bar{x})| < |\bar{y} - \bar{x}| \}, \]

so it cannot contain any other point of \( \Gamma \) different from \( x \). Thus, we can extend \( p^x \) to the whole graph \( \Gamma \) by setting \( p^x(x) = x \). Notice that \( p^x \) is a Lipschitz function (with Lipschitz constant depending on \( \text{Lip}(A) \)).

For \( y \in \Gamma \), set \( d\mu(y) := |J(A)(y)|^{-1} d\mathcal{H}^n(y) \) and \( \nu_x := p^x \mu \). Then, by the definition of \( U^{2m}_m \) in (4.1.6),

\[ U^{2m}_m(x) = \Psi DG g(x) \int_{\Gamma} K(x - y) \chi_{\epsilon_{m+1}}(x - y) d\mu(y) \]

\[ = \Psi DG g(x) \int_{\Gamma} K(x - y) \chi_{\epsilon_{m+1}}(x - y) d(\mu - \nu_x)(y) \]

\[ + \Psi DG g(x) \int_{\Gamma} K(x - y) \chi_{\epsilon_{m+1}}(x - y) d\nu_x(y) =: U^{4m}_m(x) + U^{5m}_m(x). \]  

(4.1.18)
4.1. \( L^2 \) estimates for \( \mathcal{V}_b \circ \mathcal{T}^{H^n} \) and \( \mathcal{O} \circ \mathcal{T}^{H^n} \)

Notice that \( \chi_{\epsilon_{m+1}}^\epsilon (x-y) = \chi_{\epsilon_{m+1}}^\epsilon (x - p^\xi (y)) \). This is the main reason why we use the angular projection \( p^\xi \) instead of \( p_0^\xi \) or a “vertical” one. Since \( |y - p^\xi (y)| \lesssim \text{dist}(y, L_D) \leq \text{dist}(y, L_D) + \text{dist}(x, L_D) \) for all \( y \in \Gamma \),

\[
|U_4 m(x)| \leq |\Psi_D g(x)| \int_{\Gamma} |K(x - y) - K(x - p^\xi (y))| \chi_{\epsilon_{m+1}}^\epsilon (x-y) \, d\mu(y) \\
\lesssim |\Psi_D g(x)| \ell(D)^{-n-1} \int_{\Gamma} |y - p^\xi (y)| \chi_{\epsilon_{m+1}}^\epsilon (x-y) \, d\mu(y) \\
\lesssim |\Psi_D g(x)| \ell(D)^{-n-1} \int_{\Gamma} (\text{dist}(y, L_D) + \text{dist}(x, L_D)) \chi_{\epsilon_{m+1}}^\epsilon (x-y) \, d\mu(y). \tag{4.1.19}
\]

If \( L_1^D \) denotes a minimizing \( n \)-plane for \( \beta_1(D) \), one can show that \( \text{dist}_H(L_D \cap C_T D, L_1^D \cap C_T D) \lesssim \alpha(D) \ell(D) \), so \( \text{dist}(y, L_D) \lesssim \text{dist}(y, L_1^D) + \alpha_\mu(D) \ell(D) \) for \( y \in C_T D \cap \Gamma \). Therefore,

\[
\sum_{m \in S_j} |U_4 m(x)|^2 \lesssim |\Psi_D g(x)|^2 \left( \ell(D)^{-n-1} \int_{\Gamma} (\text{dist}(y, L_D) + \text{dist}(x, L_D)) \chi_{\epsilon_{m+1}}^\epsilon (x-y) \, d\mu(y) \right)^2 \\
\lesssim |\Psi_D g(x)|^2 \left( \ell(D)^{-n-1} \int_{C_T D} (\text{dist}(y, L_D) + \text{dist}(x, L_D)) \, d\mu(y) \right)^2 \\
\lesssim |\Psi_D g(x)|^2 \left( \beta_1(D)^2 + \alpha_\mu(D)^2 + \left( \text{dist}(x, L_D) / \ell(D) \right)^2 \right). \tag{4.1.20}
\]

Let us consider \( U_5 m(x) \) now. We can assume that \( \nu_x \) is absolutely continuous with respect to \( \mathcal{H}_n^{L_D} \), because the set of points \( x \in \Gamma \) for which this statement does not hold has countable many elements, thus it has \( \mu \)-measure zero. Let \( h_x \) be the corresponding density, so \( \nu_x = h_x \mathcal{H}_n^{L_D} \). Finally, set \( u_x(y) := h_x(A_x(y))|J(A_x)| \) for \( y \in \mathbb{R}^n \). Then, since \( \int_{\mathbb{R}^n} K(x - A_x(y)) \chi_{\epsilon_{m+1}}^\epsilon (x-A_x(y)) \, dy = 0 \) because \( L_1^D \) is an \( n \)-plane containing \( x \),

\[
U_5 m(x) = \Psi_D g(x) \int_{\Gamma} K(x-y) \chi_{\epsilon_{m+1}}^\epsilon (x-y) \, d\nu_x(y) \\
= \Psi_D g(x) \int_{\mathbb{R}^n} K(x-A_x(y)) \chi_{\epsilon_{m+1}}^\epsilon (x-A_x(y)) u_x(y) \, dy \\
= \Psi_D g(x) \int_{\mathbb{R}^n} K(x-A_x(y)) \chi_{\epsilon_{m+1}}^\epsilon (x-A_x(y)) (u_x(y) - |J(A_x)|) \, dy. \tag{4.1.21}
\]

Since \( A \) has compact support, we may assume that \( h_x - 1 \in L^2(\mathcal{H}_n^{L_D}) \). Therefore, \( u_x - |J(A_x)| \in L^2(\mathcal{L}^n) \). Consider the decomposition of this function with respect to the wavelet basis, i.e., \( u_x - |J(A_x)| = \sum_{Q \in \mathcal{D}} \Delta Q (u_x - |J(A_x)|) = \sum_{Q \in \mathcal{D}} \Delta Q u_x \) (observe that, for any \( Q \in \mathcal{D} \), \( \int |J(A_x)||\psi_Q| \, d\mathcal{L}^n = 0 \)).

Set \( J := \{ Q \in \mathcal{D} : \text{supp}\psi_Q \cap \text{supp}\chi_{2^{-j-1}}^\epsilon (x-A_x(y)) \neq \emptyset \} \). Recall that \( D \in \mathcal{D}_j \) and \( m \in S_j \). Since \( x \in D \) and \( \ell(D) = 2^{-j} \), if \( Q \in J \), then \( D \subset C_b Q \) or \( Q \subset C_b D \) for \( C_b \) big
enough. In particular, if $100 \sqrt{n} C_w \ell(Q) > \ell(D)$ then $D \subset C_b Q$, and if $100 \sqrt{n} C_w \ell(Q) \leq \ell(D)$ then $Q \subset C_b D$ and $\text{dist}(x, C_w Q) \gg \ell(D)$.

We define $J_1 := \{Q \in J : 100 \sqrt{n} C_w \ell(Q) \leq \ell(D)\} \subset \{Q \in D : Q \subset C_b D\}$ and $J_2 := J \setminus J_1 \subset \{Q \in D : D \subset C_b Q\}$, thus $\text{dist}(x, C_w Q) \gg \ell(D)$ for all $Q \in J_1$. Then, since $\text{supp} \chi_{\epsilon_{m+1}}^m (x - A_x(\cdot)) \subset \text{supp} \chi_{2^{-j-1}}^m (x - A_x(\cdot))$ for all $m \in S_j$,

$$U_5 m(x) = \Psi_D g(x) \int_{\mathbb{R}^n} K(x - A_x(y)) \chi_{\epsilon_{m+1}}^m (x - A_x(y)) \sum_{Q \in J_1} \Delta Q u_x(y) dy$$

$$+ \Psi_D g(x) \int_{\mathbb{R}^n} K(x - A_x(y)) \chi_{\epsilon_{m+1}}^m (x - A_x(y)) \sum_{Q \in J_2} \Delta Q u_x(y) dy \quad (4.1.22)$$

$$=: U_6 m(x) + U_7 m(x).$$

The sum $\sum_{m \in S_j} |U_6 m(x)|^2$ can be estimated using almost the same arguments as the ones for $\sum_{m \in S_j} |U_3 m(x)|^2$ in Subsection 4.1.2.2 (see (4.1.16)), and then one obtains

$$\sum_{m \in S_j} |U_6 m(x)|^2 \lesssim |\Psi_D g(x)|^2 \sum_{Q \in J_1} \frac{\ell(Q)^{1/2}}{\ell(D)^{n+1/2}} \|\Delta Q u_x\|^2_2. \quad (4.1.23)$$

For the case of $U_7 m(x)$, recall that $\int_{\mathbb{R}^n} K(x - A_x(y)) \chi_{\epsilon_{m+1}}^m (x - A_x(y)) dy = 0$ because of the antisymmetry of $K$ and the flatness of $L_D$. Therefore,

$$U_7 m(x) = \Psi_D g(x) \sum_{Q \in J_2} \int_{\mathbb{R}^n} K(x - A_x(y)) \chi_{\epsilon_{m+1}}^m (x - A_x(y)) (\Delta Q u_x(y) - \Delta Q u_x(x)) dy,$$

and then, since $|\Delta Q u_x(y) - \Delta Q u_x(x)| \lesssim \|\Delta Q u_x\|_2 \ell(Q)^{-n/2-1} |x - y|$ by Definition 4.1.4(e),

$$|U_7 m(x)| \lesssim |\Psi_D g(x)| \sum_{Q \in J_2} \frac{\ell(D)^{-n+1}}{\ell(Q)^{-n/2-1}} \|\Delta Q u_x\|_2 \ell(Q)^{-n/2-1} \int_{\mathbb{R}^n} \chi_{\epsilon_{m+1}}^m (x - A_x(y)) dy.$$

Finally, by Cauchy-Schwarz inequality,

$$\sum_{m \in S_j} |U_7 m(x)|^2 \lesssim |\Psi_D g(x)|^2 \left( \sum_{Q \in J_2} \frac{\ell(D)^{-n+1}}{\ell(Q)^{-n/2}} \|\Delta Q u_x\|_2^2 \right)^2 \quad (4.1.24)$$

**Lemma 4.1.6.** Given $Q \in D$, one has $\|\Delta Q u_x\|_2 \lesssim \alpha_\nu(Q) \ell(Q)^{n/2}$. Moreover, there exist absolute constants $C_1, C_2 > 1$ such that, given $Q \in D$,

(a) if $D \subset C_b Q$, then

$$\alpha_\nu(Q) \lesssim \sum_{R \in D : DR \subset C_1 Q} \alpha_\mu(C_1 R) + \frac{\text{dist}(x, L_D)}{\ell(D)}, \quad \text{and}$$
4.1. $L^2$ estimates for $\mathcal{V}_R \circ T^{\mathcal{W}}_x$ and $\mathcal{O} \circ T^{\mathcal{W}}_x$

(b) if $Q \subset C_bD$ and $\text{dist}(x,C_wQ) \geq \ell(D)$, there exists a $v$-cube $Q_0 \equiv Q_0(x,Q) \in \mathcal{D}$ depending on $x$ and $Q$ such that $Q_0 \subset C_2D$, $\ell(Q_0) \approx \ell(Q)$, $\Gamma \cap Q_0 \cap (p^x)^{-1}(Q \cap L^2_D) \neq \emptyset$ and

$$\alpha_{\nu_x}(Q) \lesssim \sum_{R \in \mathcal{D}: Q_0 \subset R \subset C_2D} \alpha_{\mu}(C_2R) + \frac{\text{dist}(x,L_D)}{\ell(D)}.$$\n
**Proof.** Given $Q \in \mathcal{D}$, define the function $\phi_Q : \mathbb{R}^d \to \mathbb{R}$ by $\phi_Q(y) = \psi_Q(p_0(y))$ (see Definition 4.1.4 for $\psi_Q$). Then $\text{supp}\phi_Q \subset C_wQ$ and $\int \phi_Q d\mathcal{H}_L = 0$ for all $n$-planes $L \subset \mathbb{R}^d$ which are not perpendicular to $\mathbb{R}^n \times \{0\}^{d-n}$. Notice also that $|\nabla \phi_Q| \lesssim \ell(Q)^{-n-1}$. Let $\lambda_Q$ be a minimizing measure for $\alpha_{\nu_x}(Q)$. Then,

$$\|\Delta Q u_x\|_2 = \|\langle u_x, \psi_Q \rangle \psi_Q\|_2 = \|\langle u_x, \psi_Q \rangle\| = \left| \int_{\mathbb{R}^n} \psi_Q(y)h_x(A_x(y))|J(A_x)| \, dy \right|$$

$$\geq \left| \int_{L^2_D} \phi_Q(y) \, d\nu_x(y) \right| = \left| \int \phi_Q(y) \, d(\nu_x - \lambda_Q)(y) \right|$$

$$\lesssim \ell(Q)^{-n-1}\text{dist}_{B_Q}(\nu_x, \lambda_Q) \lesssim \ell(Q)^{n/2}\alpha_{\nu_x}(Q),$$

which proves the first statement of the lemma.

Assume now that $D \subset C_bQ$. Let $L_Q$ be a minimizing $n$-plane for $\alpha_{\mu}(C_1Q)$, where $C_1 > 1$ is some big constant to be fixed below, and let $L^x_Q$ be the $n$-plane parallel to $L_Q$ which contains $x$. Let $\sigma_Q := c_Q\mathcal{H}_L^{n}Q$ be a minimizing measure for $\alpha_{\mu}(C_1Q)$ and define $\sigma^x_Q := c_Q\mathcal{H}_L^{n}Q$. Finally, set $\sigma := c_Q\mathcal{H}_L^{n}Q$.

Similarly to what we said below (4.1.17) (recall that $\Gamma$ has slope less than 1), one can verify that $p^x$ is well defined on $L^x_Q$. Since $\sigma$ is a flat measure,

$$\alpha_{\nu_x}(Q) \leq \ell(Q)^{-n-1}\text{dist}_{B_Q}(\nu_x, \sigma) \leq \ell(Q)^{-n-1}\text{dist}_{B_Q}(\nu_x, p^x_Q\sigma_Q) + \ell(Q)^{-n-1}\text{dist}_{B_Q}(p^x_Q\sigma_Q, \sigma).$$

(4.1.25)

To deal with the first term on the right hand side of (4.1.25), let $h$ be a Lipschitz function such that $\text{supp} h \subset B_Q$ and $\text{Lip}(h) \leq 1$. Then, the function $h \circ p^x$ restricted to $\Gamma \cup L^x_Q$ can be extended to a Lipschitz function supported in $B_{C_1Q}$ (if $C_1$ is big enough) with $\text{Lip}(h \circ p^x)$ bounded by a constant which only depends on $n$, $d$, and $\text{Lip}(A)$. Therefore,

$$\left| \int_{B_Q} h(\nu_x - p^x_Q\sigma_Q) \right| = \left| \int_{B_{C_1Q}} h \circ p^x \, d(\mu - \sigma_Q) \right| \lesssim \text{dist}_{B_{C_1Q}}(\mu, \sigma_Q)$$

$$\lesssim \text{dist}_{B_{C_1Q}}(\mu, \sigma_Q) + \text{dist}_{B_{C_1Q}}(\sigma_Q, \sigma^x_Q) \lesssim \alpha_{\mu}(C_1Q)\ell(Q)^{n+1} + \text{dist}(x, L_Q)\ell(Q)^n.$$

(4.1.26)

By Remark 3.1.3 (see (3.1.7)), since $x \in \Gamma \cap D$ and $D \subset C_1Q$ (if $C_1 > C_b$),

$$\text{dist}(x, L_Q) \lesssim \sum_{R \in \mathcal{D}: D \subset R \subset C_1Q} \alpha_{\mu}(R)\ell(R) + \text{dist}(x, L_D).$$

(4.1.27)
Taking the supremum over all possible Lipschitz functions \( h \) in (4.1.26) and using that \( \ell(D) \leq \ell(R) \leq C_b \ell(Q) \) in the sum above, we get

\[
\ell(Q)^{-n-1} \text{dist}_{B_{Q}}(\nu_{x}, p_{x}^{\ell} \sigma_{Q}^{\ell}) \lesssim \sum_{R \in \mathcal{D} : \mathcal{D} \subset \mathcal{R}_{Q} \subset \mathcal{C}_{Q}} \alpha_{\mu}(C_{1}R) + \text{dist}(x, L_{D})\ell(D)^{-1}. \tag{4.1.28}
\]

To estimate the second term on the right hand side of (4.1.25), notice that \( p_{x}^{\ell} \sigma = \sigma \) because \( p^{\ell}|_{L_{D}} = \mathrm{Id} \). Hence, as in (4.1.26),

\[
dist_{B_{Q}}(p_{x}^{\ell} \sigma_{Q}^{\ell}, \sigma) = \text{dist}_{B_{Q}}(p_{x}^{\ell} \sigma_{Q}^{\ell}, p_{x}^{\ell} \sigma) \lesssim \text{dist}_{B_{C_{1}Q}}(\sigma_{Q}^{\ell}, \sigma) \leq \text{dist}_{B_{C_{1}Q}}(\sigma_{Q}^{\ell}, Q_{0}) + \text{dist}_{B_{C_{1}Q}}(\sigma_{Q}^{\ell}, \mu) \lesssim \text{dist}(x, L_{Q})\ell(Q)^{n} + \text{dist}_{B_{C_{1}Q}}(\mathcal{H}_{L_{Q}}^{n}, \mathcal{H}_{D_{Q}}^{n}) + \text{dist}(x, L_{D})\ell(Q)^{n}.
\]

The term \( \text{dist}_{B_{C_{1}Q}}(\mathcal{H}_{L_{Q}}^{n}, \mathcal{H}_{D_{Q}}^{n}) \) can be estimated using the intermediate v-cubes between \( D \) and \( C_{1}Q \) as we did in Subsection 3.3.1.2 (see (3.3.24) for example), and we obtain

\[
\text{dist}_{B_{C_{1}Q}}(\mathcal{H}_{L_{Q}}^{n}, \mathcal{H}_{D_{Q}}^{n}) \lesssim \sum_{R \in \mathcal{D} : \mathcal{D} \subset \mathcal{R}_{Q} \subset \mathcal{C}_{Q}} \alpha_{\mu}(C_{1}R)\ell(Q)^{n+1}.
\]

Thus, by (4.1.27) and since \( \ell(D) \leq \ell(Q) \),

\[
\text{dist}_{B_{Q}}(p_{x}^{\ell} \sigma_{Q}^{\ell}, \sigma) \lesssim \sum_{R \in \mathcal{D} : \mathcal{D} \subset \mathcal{R}_{Q} \subset \mathcal{C}_{Q}} \alpha_{\mu}(C_{1}R)\ell(Q)^{n+1} + \text{dist}(x, L_{D})\ell(D)^{-1}\ell(Q)^{n+1}.
\]

Then, Lemma 4.1.6(a) follows by plugging this last inequality and (4.1.28) in (4.1.25).

Let us turn our attention to Lemma 4.1.6(b), so assume that \( Q \subset C_{b}D \). Let \( C_{2} \) be some constant bigger than \( C_{b} \), and let \( Q_{0} \in \mathcal{D} \) be a minimal v-cube such that \( C_{2}Q_{0} \) contains \( \Gamma \cap (p^{\ell})^{-1}(Q \cap L_{Q}^{\ell}) \). We can assume \( Q_{0} \subset C_{2}D \) if \( C_{2} \) is big enough. We may also suppose that \( \sum_{R \in \mathcal{D} : Q_{0} \subset \mathcal{R} \subset \mathcal{C}_{2}D} \alpha_{\mu}(C_{2}R) \) is small enough, otherwise the estimate that we want to prove would be trivial; indeed, if \( 0 \leq \sum_{R \in \mathcal{D} : Q_{0} \subset \mathcal{R} \subset \mathcal{C}_{2}D} \alpha_{\mu}(C_{2}R) \) for some absolute constant \( C_{0} > 0 \), then \( \alpha_{\nu}(Q) \lesssim 1 \leq C_{0}^{-1} \sum_{R \in \mathcal{D} : Q_{0} \subset \mathcal{R} \subset \mathcal{C}_{2}D} \alpha_{\mu}(C_{2}R) \).

One can show that, if \( \sum_{R \in \mathcal{D} : Q_{0} \subset \mathcal{R} \subset \mathcal{C}_{2}D} \alpha_{\mu}(C_{2}R) \leq C_{0} \) with \( C_{0} \) small enough, then

\[
\text{diam}(\Gamma \cap (p^{\ell})^{-1}(Q \cap L_{D})^{\ell}) \lesssim \ell(Q). \tag{4.1.29}
\]

Indeed, since \( \alpha_{\mu}(C_{2}D) \) is small by our assumption, then \( \beta_{\infty, \mu}(C_{2}D) \) is also small. Take \( z_{1}, z_{2} \in \Gamma \cap (p^{\ell})^{-1}(Q \cap L_{D}^{\ell}) \) such that \( |z_{1} - z_{2}| = \text{diam}(\Gamma \cap (p^{\ell})^{-1}(Q \cap L_{D}^{\ell})) \), and set \( y_{1} := p^{\ell}(z_{1}) \) and \( y_{2} := p^{\ell}(z_{2}) \). We claim that \( |z_{1} - z_{2}| \lesssim |y_{1} - y_{2}| \). Otherwise, the angle between \( L_{z_{1}x} \) and \( L_{z_{1}z_{2}} \) would be big, where \( L_{u,v} \) denotes the line passing through the points \( u, v \in \mathbb{R}^{d} \). Since \( \beta_{\infty, \mu}(C_{2}D) \) is small by hypothesis, the angle between \( L_{z_{1}x} \) and \( L_{D}^{\ell} \) is also small. Therefore, the angle between \( L_{z_{1}z_{2}} \) and \( L_{D}^{\ell} \) would be big. Since \( \alpha_{\mu}(C_{2}Q_{0}) \) is small by hypothesis (and the same holds for \( \beta_{\infty, \mu}(C_{2}Q_{0}) \)), the angle between \( L_{z_{1}z_{2}} \) and \( L_{Q_{0}} \) is also small, where \( L_{Q_{0}} \)
is a minimizing $n$-plane for $\beta_{\infty,q}(C_2Q_0)$. Therefore, the angle between $L_{Q_0}$ and $L_D$ would be big, but this can not happen because that angle is bounded by $\sum_{R \in \mathcal{D}}d_{Q_0 \in R \subset C_2D} \alpha_\mu(C_2R)$, which is small by hypothesis. Hence, $|z_1 - z_2| \lesssim |y_1 - y_2|$, and this easily implies (4.1.29). By hypothesis, $\Gamma \cap (p')^{-1}(Q \cap L_D) \subset C_2Q_0$ and, by (4.1.29), $\ell(Q_0) \approx \ell(Q)$ if $C_0$ is small enough.

Let $L_{Q_0}$ and $\sigma_{Q_0} := c_{Q_0} \mathcal{H}^n_{L_{Q_0}}$ be a minimizing $n$-plane and measure for $\alpha_\mu(C_2Q_0)$, respectively. Fix $z_{Q_0} \in L_{Q_0} \cap B_{C_2Q_0}$ and let $L_r$ be an $n$-plane parallel to $L_D$ which contains $z_{Q_0}$. Finally, define the measures $\sigma_r := c_{Q_0} \mathcal{H}^n_{L_r}$ and $\sigma' := c_{Q_0} \mathcal{H}^n_{L'_D}$.

Notice that $p^x$ is well defined on $(L_{Q_0} \cup L_r) \cap B_{C_2Q_0}$ because $\text{dist}(x,Q) \gtrsim \ell(D)$ (we may assume that $\ell(Q)$ is small enough). Since $\sigma'$ is a flat measure, by the triangle inequality,

$$\alpha_{\nu_x}(Q)\ell(Q)^{n+1} \leq \text{dist}_{BQ}(\nu_x, \sigma') \leq \text{dist}_{BQ}(\nu_x, p^x \sigma_{Q_0}) + \text{dist}_{BQ}(p^x \sigma_{Q_0}, p^x \sigma_r) + \text{dist}_{BQ}(p^x \sigma_r, \sigma').$$

(4.1.30)

Arguing as in (4.1.26), if $C_2$ is big enough, we have

$$\text{dist}_{BQ}(\nu_x, p^x \sigma_{Q_0}) = \text{dist}_{BQ}(p^x \mu, p^x \sigma_{Q_0}) \lesssim \alpha_\mu(C_2Q_0) \ell(Q)^{n+1},$$

(4.1.31)

and

$$\text{dist}_{BQ}(p^x \sigma_{Q_0}, p^x \sigma_r) \lesssim \text{dist}_{B_{C_2Q_0}}(\sigma_{Q_0}, \sigma_r) \lesssim \text{dist}_{\mathcal{H}}(L_{Q_0} \cap B_{C_2Q_0}, L_r \cap B_{C_2Q_0})\ell(Q)^n.$$

Let $\gamma$ be the angle between $L_r$ and $L_{Q_0}$ (which is the same as the one between $L_D$ and $L_{Q_0}$). Since $z_{Q_0} \in L_{Q_0} \cap L_r \cap B_{C_2Q_0}$, we have $\text{dist}_{\mathcal{H}}(L_{Q_0} \cap B_{C_2Q_0}, L_r \cap B_{C_2Q_0}) \lesssim \sin(\gamma)\ell(Q)$, and it is not difficult to show that $\sin(\gamma) \lesssim \sum_{R \in \mathcal{D}}d_{Q_0 \in R \subset C_2D} \alpha_\mu(C_2R)$. Thus,

$$\text{dist}_{BQ}(p^x \sigma_{Q_0}, p^x \sigma_r) \lesssim \sum_{R \in \mathcal{D}}d_{Q_0 \in R \subset C_2D} \alpha_\mu(C_2R)\ell(Q)^{n+1}.$$

(4.1.32)

Let us estimate the last term on the right hand side of (4.1.30). Since $c_{Q_0} \lesssim 1$, we have

$$\text{dist}_{BQ}(p^x \sigma_r, \sigma') \lesssim \text{dist}_{BQ}(p^x \mathcal{H}^n_{L_r}, \mathcal{H}^n_{L'_D}).$$

Let $h$ be a 1-Lipschitz function supported in $B_Q$ and such that

$$\text{dist}_{BQ}(p^x \mathcal{H}^n_{L_r}, \mathcal{H}^n_{L'_D}) \approx \left| \int h d(p^x \mathcal{H}^n_{L_r} - \mathcal{H}^n_{L'_D}) \right|.$$

(4.1.33)

Set $d := \text{dist}(z_{Q_0}, L'_D)$. Without loss of generality, we may assume that $x = 0$ and that $L'_D = \mathbb{R}^n \times \{0\}^{d-n}$, so $L_r = z_{Q_0} + \mathbb{R}^n \times \{0\}^{d-n}$. Then, if we set $z'_{Q_0} := (z_{Q_0}^{n+1}, \ldots, z_{Q_0}^d)$, we have that $d = |z'_{Q_0}|$ and $p^x$ restricted to $L_r$ can be written in the following manner: $p^x : y = (y^1, \ldots, y^n, z'_{Q_0}) \rightarrow (F(y^1, \ldots, y^n), 0)$, where $F : \mathbb{R}^n \setminus \{0\}^n \rightarrow \mathbb{R}^n$ is defined by

$$F(y) = y \sqrt{|y|^2 + d^2} = y \sqrt{1 + \left| \frac{d^2}{|y|^2} \right|}.$$
Therefore, \( \int h \, d(p_x^n H_{L_x}^n) = \int h \circ p^x \, dH_{L_x}^n = \int_{\mathbb{R}^n} (h \circ p^x)(y, z_{Q_0}) \, dy = \int_{\mathbb{R}^n} h(F(y), 0) \, dy \), and we also have \( \int h \, dH_{L_D}^x = \int_{\mathbb{R}^n} h((y,0)) \, dy = \int_{\mathbb{R}^n} h(F(y), 0)J(F)(y) \, dy \) by a change of variables, where \( J(F) \) denotes the Jacobian of \( F \) (we may assume dist(0, supp\(h(F(\cdot),0)) \gtrsim \ell(D) \), because dist\((x,Q) \gtrsim \ell(D) \) and we can assume that \( \ell(Q) \) is small enough). Hence, by (4.1.33),

\[
\text{dist}_{B_0}(p_x^n H_{L_x}^n, H_{L_D}^n) \lesssim \int_{\mathbb{R}^n} |h(F(y), 0)||1 - J(F)(y)| \, dy.
\]

Notice that, because of the assumptions on supph\(F(\cdot),0) \) and since \( z_{Q_0} \in B_{C_2D} \) and \( Q_0 \subset C_2D \), we have \( d \lesssim |y| \) for all \( y \in \text{supp}h(F(\cdot),0) \). If \( F_i \) denotes the \( i \)’th coordinate of \( F \), it is straightforward to check that \( \partial_{y^i} F_i(y) = -d^2 y^i|y|^{-3}(|y|^2 + d^2)^{-1/2} \) if \( i \neq j \) and \( \partial_{y^j} F_i(y) = (1 + d^2/|y|^2)^{1/2} - d^2 |y|^{-3}(|y|^2 + d^2)^{-1/2} \). Thus, we easily obtain \( |1 - J(F)(y)| \lesssim d/|y| \lesssim d/\ell(D) \) for all \( y \in \text{supp}h(F(\cdot),0) \).

Since diam\(\text{supp}h(F(\cdot),0)) \lesssim \ell(Q) \) and \( h((F(\cdot),0)) \) is Lipschitz, using the previous comments we have \( \text{dist}_{B_0}(p_x^n H_{L_x}^n, H_{L_D}^n) \lesssim (Q)^{n+1} d/\ell(D) \). Finally, by Remark 3.1.3 (see (3.1.7)) and since \( z_{Q_0} \in L_{Q_0} \),

\[
d \lesssim \text{dist}(z_{Q_0}, L_D) + \text{dist}(L_D, L_D^x) \lesssim \sum_{R \in D: Q_0 \subset R \subset C_2D} \alpha_\mu(C_2R) \ell(R) + \text{dist}(x, L_D),
\]

and thus

\[
\text{dist}_{B_0}(p_x^n H_{L_x}^n, H_{L_D}^n) \lesssim \sum_{R \in D: Q_0 \subset R \subset C_2D} \alpha_\mu(C_2R) \ell(Q)^{n+1} + \frac{\text{dist}(x, L_D)}{\ell(D)} \ell(Q)^{n+1} \tag{4.1.34}
\]

Lemma 4.1.6(b) follows by applying (4.1.31), (4.1.32), and (4.1.34) to (4.1.30). \( \square \)

### 4.1.3 Estimate of \( \sum_{j \in \mathbb{Z}} \sum_{D \in D_j} \int_D \sum_{m \in S_j} |K_{\chi_{x_{m+1}}^e} \ast (f H_1^n)|^2 \, dH_1^n \) in (4.1.4)

From (4.1.6), (4.1.18), and (4.1.22), we have

\[
\sum_{m \in S_j} |(K_{\chi_{x_{m+1}}^e} \ast (f H_1^n))(x)|^2 \lesssim \sum_{m \in S_j} |U_{1m}(x)|^2 + |U_{3m}(x)|^2 + |U_{4m}(x)|^2 + |U_{6m}(x)|^2 + |U_{7m}(x)|^2. \tag{4.1.35}
\]

First of all, by (4.1.7) we have

\[
\sum_{j \in \mathbb{Z}} \sum_{D \in D_j} \int_D \sum_{m \in S_j} |U_{1m}|^2 \, dH_1^n \lesssim \sum_{D \in D} \sum_{Q \in D: D \subset C_2Q} \frac{\ell(D)^{n+1}}{\ell(Q)^{n+1}} \|\Delta Q g\|_2^2
\]

\[
= \sum_{Q \in D} \|\Delta Q g\|_2^2 \sum_{D \in D: D \subset C_2Q} \frac{\ell(D)^{n+1}}{\ell(Q)^{n+1}} \lesssim \sum_{Q \in D} \|\Delta Q g\|_2^2 = \|g\|_2^2,
\]
and by (4.1.16),

$$
\sum_{j \in \mathbb{Z}} \sum_{D \in \mathcal{D}_j} \int_{D} \sum_{m \in \mathcal{S}_j} |U3_m|^2 \, d\mathcal{H}^n \leq \sum_{D \in \mathcal{D}} \sum_{Q \in \mathcal{D} : Q \subset C_D} \frac{\ell(Q)^{1/2}}{\ell(D)^{1/2}} \|\Delta Q g\|^2_2 \\
= \sum_{Q \in \mathcal{D}} \|\Delta Q g\|^2_2 \sum_{D \in \mathcal{D} : Q \subset C_D} \frac{\ell(Q)^{1/2}}{\ell(D)^{1/2}} \lesssim \sum_{Q \in \mathcal{D}} \|\Delta Q g\|^2_2 = \|g\|^2_2.
$$

For the case of $U4_m(x)$, it is known that $|\Psi_D g(x)| \lesssim |g|_{C^1_D}$ for $x \in D$ (see [Da1, Part I]), where $C_a > 0$ is some constant depending on $C_b$ (see the definition of $\Psi_D g$ just after Remark 4.1.5) and $|g|_{C^1_D} := \mathcal{L}^n(C_D)^{-1} \int_{C_D} |g(y)| \, dy$. If $L_D^1$ and $L_D^2$ denote a minimizing $n$-plane for $\beta_{1,\mu}(D)$ and $\beta_{2,\mu}(D)$, respectively, one can show that $\text{dist}_{\mathcal{H}}(L_D \cap C_T D, L_D^1 \cap C_T D) \lesssim \alpha_{\mu}(D)\ell(D)$ and $\text{dist}_{\mathcal{H}}(L_D \cap C_T D, L_D^2 \cap C_T D) \lesssim \beta_{2,\mu}(D)\ell(D)$, so we have $\text{dist}(x, L_D) \lesssim \text{dist}(x, L_D^1) + \beta_{2,\mu}(D)\ell(D) + \alpha_{\mu}(D)\ell(D)$ for $x \in D \cap \Gamma$. Then, by (4.1.20),

$$
\sum_{j \in \mathbb{Z}} \sum_{D \in \mathcal{D}_j} \int_{D} \sum_{m \in \mathcal{S}_j} |U4_m|^2 \, d\mathcal{H}^n \lesssim \sum_{D \in \mathcal{D}} |g|_{C^1_D}^2 (\beta_{1,\mu}(D)^2 + \alpha_{\mu}(D)^2)\ell(D)^n + \sum_{D \in \mathcal{D}} |g|_{C^1_D}^2 \int_D \left(\frac{\text{dist}(x, L_D)}{\ell(D)}\right)^2 \, d\mathcal{H}^n(x) \\
\lesssim \sum_{D \in \mathcal{D}} |g|_{C^1_D}^2 (\beta_{1,\mu}(D)^2 + \beta_{2,\mu}(D)^2 + \alpha_{\mu}(D)^2)\ell(D)^n \lesssim \|g\|^2_2,
$$

where we used in the last inequality that the $\alpha_{\mu}$, $\beta_{1,\mu}$ and $\beta_{2,\mu}$ coefficients satisfy a Carleson packing condition, and so we can apply the Carleson’s embedding theorem.

For $U6_m(x)$, by (4.1.23) and Lemma 4.1.6(b), we have the estimate

$$
\sum_{j \in \mathbb{Z}} \sum_{D \in \mathcal{D}_j} \int_{D} \sum_{m \in \mathcal{S}_j} |U6_m|^2 \, d\mathcal{H}^n \lesssim \sum_{D \in \mathcal{D}} |g|_{C^1_D}^2 \int_D \sum_{Q \in \mathcal{D} : Q \subset C_D} \left(\frac{\ell(Q)}{\ell(D)}\right)^{n+1/2} \left(\frac{\text{dist}(x, L_D)}{\ell(D)}\right)^2 \, d\mathcal{H}^n(x) \\
+ \sum_{D \in \mathcal{D}} |g|_{C^1_D}^2 \int_D \sum_{Q \in \mathcal{D} : Q \subset C_D} \left(\frac{\ell(Q)}{\ell(D)}\right)^{n+1/2} \left(\frac{\alpha_{\mu}(C_D R)}{\ell(D)}\right)^2 \, d\mathcal{H}^n(x).
$$

Since $\sum_{D \in \mathcal{D}} |g|_{C^1_D}^2 (\ell(Q)\ell(D)^{-n})^{n+1/2} \lesssim 1$ and $\text{dist}(x, L_D) \lesssim \text{dist}(x, L_D^1) + \beta_{2,\mu}(D)\ell(D) + \alpha_{\mu}(D)\ell(D)$ for $x \in D \cap \Gamma$, the first term on the right hand side of the last inequality is bounded by

$$
\sum_{D \in \mathcal{D}} |g|_{C^1_D}^2 (\beta_{2,\mu}(D)^2 + \alpha_{\mu}(D)^2)\ell(D)^n,
$$

and hence by $C\|g\|^2_2$, by Carleson’s embedding theorem on Carleson measures. For the second term on the right side, since $\ell(Q) \approx \ell(Q_0(x, Q))$ (recall the definition of $Q_0 \equiv Q_0(x, Q)$ in Lemma 4.1.6(b)), $Q_0(x, Q) \subset C_2 D$, and every $Q_0 \in \mathcal{D}$ intersects $\Gamma \cap (p^x)^{-1}(Q \cap L_D^2)$ for finitely many $v$-cubes $Q \in \mathcal{D}$ (with a bound for the number of such $v$-cubes $Q$ independent
of $x$ and $Q_0$), we have
\[
\sum_{Q \in D: Q \subset C_6 D} \left( \frac{\ell(Q)}{\ell(D)} \right)^{n+1/2} \left( \sum_{R \in D: Q_0(x, Q) \subset R \subset C_2 D} \alpha_\mu(C_2 R) \right)^2 \\
= \sum_{S \in D: S \subset C_2 D} \sum_{Q \in D: Q \subset D, Q_0(x, Q) = S} \left( \frac{\ell(Q)}{\ell(D)} \right)^{n+1/2} \left( \sum_{R \in D: S \subset R \subset C_2 D} \alpha_\mu(C_2 R) \right)^2 \\
\lesssim \sum_{S \in D: S \subset C_2 D} \left( \frac{\ell(S)}{\ell(D)} \right)^{n+1/2} \left( \sum_{R \in D: S \subset R \subset C_2 D} \alpha_\mu(C_2 R) \right)^2.
\]

By Cauchy-Schwarz inequality,
\[
\sum_{S \in D: S \subset C_2 D} \left( \frac{\ell(S)}{\ell(D)} \right)^{n+1/2} \left( \sum_{R \in D: S \subset R \subset C_2 D} \alpha_\mu(C_2 R) \right)^2 \\
\lesssim \sum_{S \in D: S \subset C_2 D} \left( \frac{\ell(S)}{\ell(D)} \right)^{n+1/4} \log_2 \left( \frac{\ell(D)}{\ell(S)} \right) \sum_{R \in D: S \subset R \subset C_2 D} \alpha_\mu(C_2 R)^2 \\
\lesssim \sum_{R \in D: R \subset C_2 D} \alpha_\mu(C_2 R)^2 \sum_{S \in D: S \subset R} \left( \frac{\ell(S)}{\ell(D)} \right)^{n+1/4} \\
\lesssim \sum_{R \in D: R \subset C_2 D} \alpha_\mu(C_2 R)^2 \left( \frac{\ell(R)}{\ell(D)} \right)^{n+1/4} =: \lambda_1(D)^2.
\]

Therefore,
\[
\sum_{D \in D} \left| g \right|_{C_n D}^2 \int_D \sum_{Q \in D: Q \subset C_6 D} \left( \frac{\ell(Q)}{\ell(D)} \right)^{n+1/2} \left( \sum_{R \in D: Q_0(x, Q) \subset R \subset C_2 D} \alpha_\mu(C_2 R) \right)^2 d\mathcal{H}_n^r(x) \\
\lesssim \sum_{D \in D} \left| g \right|_{C_n D}^2 \lambda_1(D)^2 \ell(D)^n.
\]

Let us check that the coefficients $\lambda_1(D)$ satisfy a Carleson packing condition, so they originate a Carleson measure. For all $S \in D$,
\[
\sum_{D \in D: D \subset S} \lambda_1(D)^2 \ell(D)^n = \sum_{D \in D: D \subset S} \sum_{R \in D: R \subset C_2 D} \alpha_\mu(C_2 R)^2 \left( \frac{\ell(R)}{\ell(D)} \right)^{n+1/4} \ell(D)^n \\
\leq \sum_{R \in D: R \subset C_2 S} \alpha_\mu(C_2 R)^2 \ell(R)^n \sum_{D \in D: R \subset C_2 D} \left( \frac{\ell(R)}{\ell(D)} \right)^{1/4} (4.1.36) \\
\lesssim \sum_{R \in D: R \subset C_2 S} \alpha_\mu(C_2 R)^2 \ell(R)^n \lesssim \ell(S)^n.
\]
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Then, $\sum_{j \in \mathbb{Z}} \sum_{D \in D_j} \int_D \sum_{m \in S_j} |U6_m|^2 \, d\mathcal{H}^n_\Gamma \lesssim \|g\|_2^2$, by the Carleson embedding theorem.

For $U7_m(x)$, using (4.1.24) and Lemma 4.1.6(a), we have

$$
\sum_{j \in \mathbb{Z}} \sum_{D \in D_j} \int_D \sum_{m \in S_j} |U7_m|^2 \, d\mathcal{H}^n_\Gamma \lesssim \sum_{D \in D} |g|^2_{C_a D} \int_D \sum_{Q \in D : D \subset C_d Q} \frac{\ell(D)}{\ell(Q)} \left( \frac{\text{dist}(x, L_D)}{\ell(D)} \right)^2 \, d\mathcal{H}^n_\Gamma(x)
+ \sum_{D \in D} |g|^2_{C_a D} \ell(D)^n \sum_{Q \in D : D \subset C_d Q} \frac{\ell(D)}{\ell(Q)} \sum_{R \in D : D \subset R \subset C_d Q} \alpha_\mu(C_1 R)^2.
$$

Since $\text{dist}(x, L_D) \leq \text{dist}(x, L_D^2) + \beta_2 \mu(D) \ell(D) + \alpha_\mu(D) \ell(D)$ for $x \in D \cap \Gamma$, by Cauchy-Schwarz and Carleson’s embedding theorem for the $\beta_2 \mu$’s and $\alpha_\mu$’s,

$$
\sum_{j \in \mathbb{Z}} \sum_{D \in D_j} \int_D \sum_{m \in S_j} |U7_m|^2 \, d\mathcal{H}^n_\Gamma \lesssim \sum_{D \in D} |g|^2_{C_a D} \int_D \left( \frac{\text{dist}(x, L_D)}{\ell(D)} \right)^2 \, d\mathcal{H}^n_\Gamma(x)
+ \sum_{D \in D} |g|^2_{C_a D} \ell(D)^n \sum_{Q \in D : D \subset C_d Q} \frac{\ell(D)}{\ell(Q)} \sum_{R \in D : D \subset R \subset C_d Q} \alpha_\mu(C_1 R)^2
\lesssim \|g\|_2^2 + \sum_{D \in D} |g|^2_{C_a D} \ell(D)^n \sum_{Q \in D : D \subset C_d Q} \frac{\ell(D)}{\ell(Q)} \sum_{R \in D : D \subset R \subset C_d Q} \alpha_\mu(C_1 R)^2
\lesssim \|g\|_2^2 + \sum_{D \in D} |g|^2_{C_a D} \ell(D)^n \sum_{R \in D : D \subset R} \alpha_\mu(C_1 R)^2 \left( \frac{\ell(D)}{\ell(R)} \right)^{1/2}.
$$

We are going to check that the coefficients $\lambda_2(D)^2 := \sum_{R \in D : D \subset R} \alpha_\mu(C_1 R)^2 \left( \frac{\ell(D)}{\ell(R)} \right)^{1/2}$ satisfy a Carleson packing condition, so they provide a Carleson measure; and then we will conclude that $\sum_{j \in \mathbb{Z}} \sum_{D \in D_j} \int_D \sum_{m \in S_j} |U7_m|^2 \, d\mathcal{H}^n_\Gamma \lesssim \|g\|_2^2$. For all $S \in D$, we have

$$
\sum_{D \in D : D \subset S} \lambda_2(D)^2 \ell(D)^n \sum_{D \in D : D \subset DCS} \sum_{R \in D : D \subset DCR} \alpha_\mu(C_1 R)^2 \left( \frac{\ell(D)}{\ell(R)} \right)^{1/2} \ell(D)^n
= \sum_{D \in D : D \subset S} \sum_{R \in D : D \subset DCR} \alpha_\mu(C_1 R)^2 \left( \frac{\ell(D)}{\ell(R)} \right)^{n+1/2} \ell(R)^n
+ \sum_{D \in D : D \subset S} \sum_{R \in D : D \subset S \subset R} \alpha_\mu(C_1 R)^2 \left( \frac{\ell(D)}{\ell(R)} \right)^{n+1/2} \ell(R)^n
=: I + II.
$$

Concerning $I$, since the $\alpha_\mu$’s satisfy a Carleson packing condition, we get

$$
I = \sum_{R \in D : R \subset S} \alpha_\mu(C_1 R)^2 \ell(R)^n \sum_{D \in D : D \subset R} \left( \frac{\ell(D)}{\ell(R)} \right)^{n+1/2} \lesssim \sum_{R \in D : R \subset S} \alpha_\mu(C_1 R)^2 \ell(R)^n \lesssim \ell(S)^n,
$$
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as wished. For the case of II we use the estimate \( \alpha_\mu(C_1 R) \lesssim 1 \) for all \( R \in \mathcal{D} \), thus

\[
II \lesssim \sum_{R \in \mathcal{D} : S \subseteq R} \sum_{D \in \mathcal{D} : D \subseteq S} \left( \frac{\ell(D)}{\ell(R)} \right)^{n+1/2} \ell(R)^n = \sum_{R \in \mathcal{D}_j : S \subseteq R} \left( \frac{\ell(S)}{\ell(R)} \right)^{n+1/2} \ell(R)^n \sum_{D \in \mathcal{D}_j : D \subseteq S} \left( \frac{\ell(D)}{\ell(S)} \right)^{n+1/2} \ell(S)^n.
\]

Therefore, \( \sum_{D \in \mathcal{D}_j : D \subseteq S} \lambda_2(D)^2 \ell(D)^n \lesssim \ell(S)^n \), as claimed.

Finally, plugging all these estimates in (4.1.35), we conclude that

\[
\sum_{j \in \mathbb{Z}} \sum_{D \in \mathcal{D}_j} \int_D \sum_{m \in \mathcal{S}_j} |K \chi_{\kappa_m} \ast (f \mathcal{H}_\Gamma^n)|^2 d\mathcal{H}_\Gamma^n \lesssim \|g\|_2^2 \approx \|f\|_{L^2(\mathcal{H}_\Gamma^n)}^2,
\]

as desired.

4.1.4 Estimate of \( \sum_{j \in \mathbb{Z}} \sum_{D \in \mathcal{D}_j} \int_D \sum_{m \in \mathcal{L} : \kappa_m \in I_j} |K \kappa_m \ast (f \mathcal{H}_\Gamma^n)|^2 d\mathcal{H}_\Gamma^n \) in (4.1.4)

Recall that, for \( \epsilon > 0 \), we have set \( \kappa_\epsilon := \chi_\epsilon - \tilde{\varphi}_\epsilon \) (see the line before (4.1.3)).

The arguments to estimate \( \sum_{j \in \mathbb{Z}} \sum_{D \in \mathcal{D}_j} \int_D \sum_{m \in \mathcal{L} : \kappa_m \in I_j} |K \kappa_m \ast (f \mathcal{H}_\Gamma^n)|^2 d\mathcal{H}_\Gamma^n \) are very similar to the ones in the previous subsections. Basically, we have to replace the function \( \chi_{\kappa_m} \) by \( \kappa_m \) in all the proofs in Subsections 4.1.2 and 4.1.3, because, in most of the estimates, we only used the properties of the support and the symmetry of the function \( \chi_{\kappa_m} \), and \( \kappa_m \) satisfies analogous properties (\( \kappa_\epsilon \) is supported in the closure of \( (B^n(0, 3\sqrt{n} \epsilon) \times \mathbb{R}^{d-n}) \) \( B^d(0, \epsilon) \subset \mathbb{R}^d \)). Notice that the sum \( \sum_{m \in \mathcal{L} : \kappa_m \in I_j} |(K \kappa_m \ast (f \mathcal{H}_\Gamma^n))(x)|^2 \) only has one term (or none) for each \( x \in \Gamma \) and \( j \in \mathbb{Z} \).

There are only two details that have to be pointed. The first one is in equation (4.1.12). Instead, now we have

\[
U^3_m(x, Q) = \int \kappa_m(x - A_\Gamma(y)) \left( K(x - A_\Gamma(y)) - K(x - A_\Gamma(z_Q)) \right) \Delta_Q g(y) \, dy
\]

\[
+ \int \left( \kappa_m(x - A_\Gamma(y)) - \kappa_m(x - A_\Gamma(z_Q)) \right) K(x - A_\Gamma(z_Q)) \Delta_Q g(y) \, dy
\]

\[
= \int \kappa_m(x - A_\Gamma(y)) \left( K(x - A_\Gamma(y)) - K(x - A_\Gamma(z_Q)) \right) \Delta_Q g(y) \, dy
\]

\[
+ \int \left( \chi_m(x - A_\Gamma(y)) - \chi_m(x - A_\Gamma(z_Q)) \right) K(x - A_\Gamma(z_Q)) \Delta_Q g(y) \, dy
\]

\[
+ \int \left( \tilde{\varphi}_m(x - A_\Gamma(z_Q)) - \tilde{\varphi}_m(x - A_\Gamma(y)) \right) K(x - A_\Gamma(z_Q)) \Delta_Q g(y) \, dy
\]

\[
=: U^3_m^A(x, Q) + U^3_m^B(x, Q) + U^3_m^C(x, Q).
\]
The terms $U^A_m(x, Q)$ and $U^B_m(x, Q)$ can be handled as above, at the end of Subsection 4.1.2.2, and the term $U^C_m(x, Q)$ can be easily estimated using the smoothness of $\bar{\varphi}_{\epsilon_m}$. Indeed, notice that $|\nabla \bar{\varphi}_{\epsilon_m}| \lesssim \ell(D)^{-1}$ for $\epsilon_m \in I_j$ and $D \in D_j$. Therefore, $|U^C_m(x, Q)| \lesssim \ell(Q) \ell(D)^{-n-1}\|\Delta_qg\|_1$, and then one can continue with the same arguments as when we estimated $U^A_m(x, Q)$ (see the equation before (4.1.13)).

The other point that has to be mentioned is in (4.1.18). Instead, now we have

$$U^2_m(x) = \Psi_D g(x) \int_G K(x - y)\kappa_{\epsilon_m}(x - y) d\mathcal{H}^n(y)$$

$$= \Psi_D g(x) \int_G K(x - y)\kappa_{\epsilon_m}(x - y) d(\mathcal{H}^n - \nu_x)(y)$$

$$+ \Psi_D g(x) \int_G K(x - y)\kappa_{\epsilon_m}(x - y) d\nu_x(y) =: U^4_m(x) + U^5_m(x).$$

The term $U^5_m(x)$ can be handled as we did previously, in the case of the function $\chi_{\epsilon_{m+1}}^m$ (see (4.1.21) and the subsequent arguments). To deal with $U^4_m(x)$, using that $\chi_{\epsilon_m}(x - p^r(y)) = \chi_{\epsilon_m}(x - y)$, we obtain

$$U^4_m(x) = \Psi_D g(x) \int_G K(x - y)\kappa_{\epsilon_m}(x - y) d(\mu - \nu_x)(y)$$

$$= \Psi_D g(x) \int_G \left( K(x - y)\kappa_{\epsilon_m}(x - y) - K(x - p^r(y))\kappa_{\epsilon_m}(x - p^r(y)) \right) d\mu(y)$$

$$= \Psi_D g(x) \int_G \left( K(x - y) - K(x - p^r(y)) \right) \kappa_{\epsilon_m}(x - y) d\mu(y)$$

$$+ \Psi_D g(x) \int_G \left( K(x - p^r(y))\bar{\varphi}_{\epsilon_m}(x - p^r(y)) - K(x - y)\bar{\varphi}_{\epsilon_m}(x - y) \right) d\mu(y)$$

$$= \Psi_D g(x) \int_G \left( K(x - y) - K(x - p^r(y)) \right) \kappa_{\epsilon_m}(x - y) d\mu(y)$$

$$+ \Psi_D g(x) \int_G K(x - p^r(y))\left( \bar{\varphi}_{\epsilon_m}(x - p^r(y)) - \bar{\varphi}_{\epsilon_m}(x - y) \right) d\mu(y).$$

The first term on the right hand side of the last equality can be handled as we did for the case of the function $\chi_{\epsilon_{m+1}}^m$ (see (4.1.19)). The second term can be easily estimated by $|\Psi_D g(x)|(|\beta_{1,D}(D) + \text{dist}(x, L_D))\ell(D)^{-1}$, using the smoothness of $\bar{\varphi}_{\epsilon_m}$ and that $|y - p^r(y)| \lesssim \text{dist}(y, L_D) \leq \text{dist}(y, L_D) + \text{dist}(x, L_D)$ for all $y \in G$. Therefore, (4.1.20) still holds replacing $\chi_{\epsilon_{m+1}}^m$ by $\kappa_{\epsilon_m}$.

### 4.1.5 Estimate of $\sum_{j \in \mathbb{Z}} \sum_{D \in D_j} \int_D \sum_{m \in \mathcal{L}} : \epsilon_{m+1} \in I_j |K\kappa_{\epsilon_{m+1}} * (f\mathcal{H}^n)|^2 d\mathcal{H}^n$ in (4.1.4)

One argues exactly as in Subsection 4.1.4 and obtains the same estimates.
Remark 4.1.7. By easier arguments one can also show that the operators $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{H\ell}$ and $\mathcal{O} \circ \mathcal{T}_\varphi^{H\ell}$ are bounded in $L^2(\mathcal{H}^n_\Gamma)$. To estimate these operators, one does not need to introduce the angular projection $p^\varphi$ that we used in the previous subsections. It is enough to use vertical projections, as in the case of $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{H\ell}$ in Sections 3.3 and 3.4. These projections behave well with respect to the truncations $\chi$. Furthermore, as we remarked at the beginning of Section 4.1, the use of Lemma 4.1.2 is not necessary and the $L^2$ boundedness holds for any Lipschitz graph $\Gamma$ (i.e. for any Lip(A) < \infty).

The $L^2$ boundedness of the operators $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{H\ell}$ and $\mathcal{O} \circ \mathcal{T}_\varphi^{H\ell}$ is easier to obtain than in the case $\omega \in \{\chi, \tilde{\chi}\}$ (using similar arguments), because now the difficult parts (which were the ones involving differences of characteristic functions) are estimated using the regularity of the functions $\varphi$ for $\rho > 0$, as in Section 3.6 with the truncations $\tilde{\varphi}_\rho$. Once we know the $L^2$ boundedness of $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{H\ell}$ and $\mathcal{O} \circ \mathcal{T}_\varphi^{H\ell}$, we can argue as in Subsection 3.6.1 to prove that these operators are also bounded from $H^1(\mathcal{H}^n_\Gamma)$ to $L^1(\mathcal{H}^n_\Gamma)$, because the $L^2$ boundedness implies the local $L^2$ estimates of Section 3.5.

4.1.6 Proof of Lemma 4.1.2

We need the following auxiliary result:

Lemma 4.1.8. Let $0 < n < d$. For $x := (x_1, \ldots, x_d) \in \mathbb{R}^d$ we denote

$$x_H := (x_1, \ldots, x_n, 0, \ldots, 0) \in \mathbb{R}^d \quad \text{and} \quad x_V := (0, \ldots, 0, x_{n+1}, \ldots, x_d) \in \mathbb{R}^d.$$ 

Given $x, y \in \mathbb{R}^d \setminus \{0\}$, if there exists $0 < s < 1$ such that $|x_V| \leq s|x_H|$, $|y_V| \leq s|y_H|$, and $|x_V - y_V| \leq s|x_H - y_H|$, then there exists $C > 0$ depending only on $s$ such that

$$|x_V - y_V| \leq C \left| |x||x_H|^{-1}x_H - |y||y_H|^{-1}y_H \right|.$$  \hspace{1cm} (4.1.38)

Proof. We set $\Phi(x, y) := \left| |x||x_H|^{-1}x_H - |y||y_H|^{-1}y_H \right|$. Since $\Phi$ is symmetric in $x$ and $y$, we can assume that $|x_H| \leq |y_H|$. If $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^d$, using the polarization identity,

$$\Phi(x, y)^2 = |x|^2 + |y|^2 - 2|x||x_H|^{-1}|y||y_H|^{-1}\langle x, y_H \rangle$$

$$= |x|^2 + |y|^2 + |x||x_H|^{-1}|y||y_H|^{-1}( |x_H - y_H|^2 - |x_H|^2 - |y_H|^2 )$$

$$= |x|^2 + |y|^2 - 2|x||y| + |x||x_H|^{-1}|y||y_H|^{-1}( |x_H - y_H|^2 - |x_H|^2 - |y_H|^2 + 2|x_H||y_H| )$$

$$= (|x| - |y|)^2 + |x||x_H|^{-1}|y||y_H|^{-1}( |x_H - y_H|^2 - (|x_H| - |y_H|)^2 ) \).$$

Since $|x_H - y_H|^2 - (|x_H| - |y_H|)^2 \geq 0$, $|x_H| \leq |x|$, and $|y_H| \leq |y|$, we have

$$\Phi(x, y)^2 \geq (|x| - |y|)^2 + |x_H - y_H|^2 - (|x_H| - |y_H|)^2.$$  \hspace{1cm} (4.1.39)
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Assume that $2|x| \leq |y|$. Then, using (4.1.39),

$$|x_V - y_V| \leq |x| + |y| \leq \frac{3}{2}|y| = 3\left(|y| - \frac{1}{2}|y|\right) \leq 3(|y| - |x|) \leq 3\Phi(x, y),$$

and we obtain (4.1.38). By the same arguments, if $2|y| \leq |x|$, then $|x_V - y_V| \leq 3\Phi(x, y)$ and (4.1.38) holds. Thus, from now on we assume $\frac{1}{2}|x| \leq |y| \leq 2|x|$.

Let $0 < \delta < 1$ be a small number that will be fixed below. Assume that $(1 - \delta)|x_H - y_H| \geq ||y_H| - |x_H||$. Then, by (4.1.39),

$$\Phi(x, y)^2 \geq |x_H - y_H|^2 - (|x_H| - |y_H|)^2 \geq |x_H - y_H|^2 - (1 - \delta)^2|x_H - y_H|^2 \geq \delta(2 - \delta)|x_H - y_H|^2 \geq \delta(2 - \delta)s^{-2}|x_V - y_V|^2,$$

and then (4.1.38) holds with $C = s/\sqrt{2(2 - \delta)}$.

Therefore, we can suppose that $(1 - \delta)|x_H - y_H| \leq ||y_H| - |x_H|| = |y_H| - |x_H|$, since we are also assuming $|x_H| \leq |y_H|$. If we set $z = y - x$, we have $(1 - \delta)|z_H| \leq |x_H + z_H| - |x_H|$, so $(1 - \delta)|z_H| + |x_H| \leq |x_H + z_H|$. Hence,

$$(1 - \delta)^2|z_H|^2 + |x_H|^2 + 2(1 - \delta)|z_H||x_H| = (1 - \delta)|z_H| + |x_H|^2$$

and we obtain

$$\langle x_H, z_H \rangle \geq -\frac{1}{2} \delta(2 - \delta)|z_H|^2 + (1 - \delta)|z_H||x_H|. \quad (4.1.40)$$

Using (4.1.40), that $\langle x_V, z_V \rangle \geq -|x_V||z_V|$, and that $|x_V| \leq s|x_H|$ and $|z_V| \leq s|z_H|$, we get

$$\langle x, z \rangle = \langle x_H + x_V, z_H + z_V \rangle = \langle x_H, z_H \rangle + \langle x_V, z_V \rangle$$

$$\geq -\frac{1}{2} \delta(2 - \delta)|z_H|^2 + (1 - \delta)|z_H||x_H| - |x_V||z_V| \quad (4.1.41)$$

$$\geq -\frac{1}{2} \delta(2 - \delta)|z_H|^2 + (1 - \delta - s^2)|z_H||x_H|.$$ 

Notice that, if $\delta > 0$ is small enough depending on $s$, then $-\frac{1}{4} (1 - s^2)(1 + s^2)^{-1} < -\frac{3}{2} \delta(2 - \delta) < 0$ and $1 - \delta - s^2 > \frac{1}{2} (1 - s^2)$. Let $\gamma(x, z)$ be the angle between $x$ and $z$ (by definition, $0 \leq \gamma(x, z) \leq \pi$). Using that $\langle x, z \rangle = |x||z| \cos(\gamma(x, z))$, that $|x| \leq \sqrt{1 + s^2}|x_H|$ and $|z| \leq \sqrt{1 + s^2}|z_H|$, and that $|z| \leq |x| + |y| \leq 3|x|$, we finally obtain from (4.1.41) that

$$\cos(\gamma(x, z)) \geq -\frac{1}{2} \delta(2 - \delta)|z_H|^2|x|^{-1}|z|^{-1} + (1 - \delta - s^2)|z_H||x_H||x|^{-1}|z|^{-1}$$

$$\geq -\frac{3}{2} \delta(2 - \delta) + (1 - \delta - s^2)(1 + s^2)^{-1} \geq \frac{1}{4} (1 - s^2)(1 + s^2)^{-1} =: a.$$

Notice that $a > 0$, because $0 < s < 1$ by hypothesis. Hence, since $\cos(\gamma(-x, y - x)) = \cos(\gamma(-x, z)) = -\cos(\gamma(x, z))$ (because $z = y - x$ and $\langle -x, z \rangle = -\langle x, z \rangle$), we have $c_0 :=$
\[ \cos(\gamma(-x, y, -x)) \leq -a < 0 \] (notice that \( c_0 \leq 0 \) implies that \( |x| \leq |y| \)). By the cosines theorem, \( |y|^2 = |x|^2 - |y - x|^2 - 2|x||y - x|c_0 \). Since \( c_0 < 0 \), we solve the second degree equation in \( |y - x| \) and obtain

\[
|y - x| = \frac{\sqrt{|y|^2 - |x|^2(1 - c_0^2)} - |x||c_0|}{\sqrt{|y|^2 - |x|^2(1 - c_0^2)} + |x||c_0|} \leq \frac{(|y| - |x|)(|y| + |x|)}{|x||c_0|} \leq \left( |y| - |x| \right) \frac{3}{a},
\]

where we also used that \( |y| \leq 2|x| \) in the last inequality. Therefore, by (4.1.39),

\[
|x_V - y_V| \leq |x - y| \leq \frac{3}{a} \left( |y| - |x| \right) \leq \frac{3}{a} \Phi(x, y),
\]

and (4.1.38) follows with \( C = 3/a \), where \( a > 0 \) only depends on \( s \). This completes the proof of the lemma.

Let us recall Lemma 4.1.2 and let us prove it:

**Lemma 4.1.9.** Let \( \Gamma := \{ x \in \mathbb{R}^d : x = (x, A(x)) \} \) be the graph of a Lipschitz function \( A : \mathbb{R}^n \to \mathbb{R}^{d-n} \) such that \( \text{Lip}(A) < 1 \). Then, \( \mathcal{H}^n_\Gamma(A^d(z, a, b)) \lesssim (b - a)b^{n-1} \) for all \( 0 < a \leq b \) and \( z \in \Gamma \).

**Proof.** We keep the notation introduced in Lemma 4.1.8. Fix \( z \in \Gamma \). We can assume that \( z = 0 \), by taking a translation of \( \Gamma \) if it is necessary.

For \( x \in \mathbb{R}^d \) with \( x_H \neq 0 \), consider the map

\[
\Upsilon(x) := \frac{|x|}{|x_H|} x_H + x_V = \sqrt{1 + \frac{|x_V|^2}{|x_H|^2}} x_H + x_V.
\]

It is not difficult to show that \( \Upsilon \) is a bilipschitz mapping from (a neighborhood of) the cone \( L := \{ x \in \mathbb{R}^d \setminus \{0\} : |x_V| \leq \text{Lip}(A)|x_H| \} \) to (a neighborhood of) the cone \( L' := \{ x \in \mathbb{R}^d \setminus \{0\} : |x_V| \leq \text{Lip}(A)(1 + \text{Lip}(A)^2)^{-1/2}|x_H| \} \), whose inverse equals

\[
\Upsilon^{-1}(x) = \sqrt{1 - \frac{|x_V|^2}{|x_H|^2}} x_H + x_V.
\]

Moreover, when \( \Upsilon \) and \( \Upsilon^{-1} \) are restricted to \( L \) and \( L' \) respectively, \( \text{Lip}(\Upsilon) \) and \( \text{Lip}(\Upsilon^{-1}) \) only depend on \( n, d \), and \( \text{Lip}(A) \). Hence, since \( \Gamma \subset L \cup \{0\} \), for any \( 0 < a \leq b \) we have

\[
\mathcal{H}^n_\Gamma(A(0, a, b)) = \mathcal{H}^n_\Gamma(\Gamma \cap A(0, a, b)) \approx \mathcal{H}^n_\Gamma(\Upsilon(\Gamma \cap A(0, a, b))).
\]

Consider the set \( \Upsilon(\Gamma) \). Since \( \Gamma \) has slope smaller than 1 (i.e. \( \text{Lip}(A) < 1 \)), by Lemma 4.1.8 there exists a constant \( C > 0 \) depending only on \( n, d \), and \( \text{Lip}(A) \) such that for any two points
4.2. $L^p$ and endpoint estimates for $\mathcal{V}_\rho \circ \mathcal{T}_\chi^{H^n_1}$ and $\mathcal{O} \circ \mathcal{T}_\chi^{H^n_1}$

$x, y \in \mathcal{Y}(\Gamma)$ one has $|x_V - y_V| \leq C|x_H - y_H|$. Then, it is known that $\mathcal{Y}(\Gamma)$ is contained in the $n$-dimensional graph $\Gamma'$ of some Lipschitz function (see for example the proof of [Ma1, Lemma 15.13]). Notice also that, given $0 < a \leq b$, $\mathcal{Y}(L \cap A(0, a, b)) \subset \{ x \in \mathbb{R}^d : a \leq |x_H| \leq b \}$. Therefore,

$$\mathcal{H}^n_1(A(0, a, b)) \approx \mathcal{H}^n(\mathcal{Y}(\Gamma \cap A(0, a, b))) \leq \mathcal{H}^n(\Gamma' \cap \{ x \in \mathbb{R}^d : a \leq |x_H| \leq b \}) \lesssim (b - a)b^{n-1},$$

and the lemma is proved. \hfill $\square$

**Remark 4.1.10.** With a little more of effort, one can show that $\mathcal{Y}(\Gamma)$ is a Lipschitz graph.

**Remark 4.1.11.** Lemma 4.1.2 is sharp in the sense that the estimate fails if $\text{Lip}(A) \geq 1$ (notice that the constant $C$ in Lemma 4.1.8 is bigger than $(1+\text{Lip}(A)^2)/(1-\text{Lip}(A)^2)$). Given $\epsilon > 0$, one can easily construct a Lipschitz graph $\Gamma$ such that $1 < \text{Lip}(A) < 1+\epsilon$ and such that, for some $z \in \Gamma$ and $r > 0$, $\Gamma$ contains a set $P \subset \partial B(z, r)$ with $\mathcal{H}^n_1(P) > 0$. Then, if Lemma 4.1.2 were true for $\Gamma$, we would have $0 < \mathcal{H}^n_1(P) \leq \mathcal{H}^n_1(A(z, r - \delta, r + \delta)) \lesssim 2\delta(r + \delta)^{n-1}$, and we would have a contradiction by making $\delta \to 0$. By a similar argument, one can also show that the lemma fails in the limiting case $\text{Lip}(A) = 1$.

4.2 $L^p$ and endpoint estimates for $\mathcal{V}_\rho \circ \mathcal{T}_\chi^{H^n_1}$ and $\mathcal{O} \circ \mathcal{T}_\chi^{H^n_1}$

**Theorem 4.2.1.** Let $\rho > 2$ and assume that $\text{Lip}(A) < 1$. The operators $\mathcal{V}_\rho \circ \mathcal{T}_\chi^{H^n_1}$ and $\mathcal{O} \circ \mathcal{T}_\chi^{H^n_1}$ are bounded

- in $L^p(\mathcal{H}^n_1)$ for $1 < p < \infty$,
- from $L^1(\mathcal{H}^n_1)$ to $L^{1,\infty}(\mathcal{H}^n_1)$, and
- from $L^{\infty}(\mathcal{H}^n_1)$ to $BMO(\mathcal{H}^n_1)$,

and the norm of $\mathcal{O} \circ \mathcal{T}_\chi^{H^n_1}$ in the cases above is bounded independently of the sequence that defines the oscillation.

We will only give the proof of Theorem 4.2.1 in the case of the $\rho$-variation, because the proof for the oscillation follows by very close arguments.

4.2.1 The operator $\mathcal{V}_\rho \circ \mathcal{T}_\chi^{H^n_1} : L^1(\mathcal{H}^n_1) \to L^{1,\infty}(\mathcal{H}^n_1)$ is bounded

By Theorem 4.1.1, we know that $\mathcal{V}_\rho \circ \mathcal{T}_\chi^{H^n_1} : L^2(\mathcal{H}^n_1) \to L^2(\mathcal{H}^n_1)$ is bounded. In Theorem B of [CJR2] it is proved that, if the variation for a singular integral in $\mathbb{R}^n$ with respect to the measure $\mathcal{L}^n$ is a bounded operator in $L^2$ and the kernel satisfies standard estimates, then the variation is also bounded from $L^1$ to $L^{1,\infty}$. Because of the AD regularity of the measure $\mathcal{H}^n_1$, it is not difficult to adapt Theorem B of [CJR2] to our setting (i.e., when the space is not $\mathbb{R}^n$ but an $n$-dimensional Lipschitz graph) by using Lemma 4.1.2, and then the weak-$L^1$ estimate follows. For more details see Theorem 5.0.12(b).
4. Variation for singular integrals on Lipschitz graphs: Rough truncation

4.2.2 The operator $\mathcal{V}_\rho \circ T^H_\chi : L^\infty(\mathcal{H}_1^n) \to BMO(\mathcal{H}_1^n)$ is bounded

The arguments are very similar to the ones in Subsection 3.6.2, and we will use analogous techniques and notation (replacing $\tilde{\varphi}$ by $\chi$). We set $f_1 := f\chi_D$ and $f_2 := f - f_1$. The case of $f_1$ is handled as in Subsection 3.6.2, but replacing Theorem 3.5.1 by Theorem 4.1.1. In the case of $f_2$, for $x \in \Gamma \cap D$, we decompose

$$|(\mathcal{V}_\rho \circ T^H_\chi)f_2(x) - c|^p \lesssim \|f\|_{L^\infty(\mathcal{H}_1^n)}^p \sum_{m \in \mathbb{Z}} (\Theta_{1m} + \Theta_{2m})^p,$$

where $c := (\mathcal{V}_\rho \circ T^H_\chi)f_2(z_D)$, and

$$\Theta_{1m} := \int_{(3D)c} \chi_{\epsilon_{m+1}}(x - y) |K(x - y) - K(z_D - y)| \, d\mathcal{H}_1^n(y),$$

$$\Theta_{2m} := \int_{(3D)c} |\chi_{\epsilon_{m+1}}(x - y) - \chi_{\epsilon_{m}}(z_D - y)||K(z_D - y)| \, d\mathcal{H}_1^n(y).$$

Arguing as in Subsection 3.6.2, we have

$$\left( \sum_{m \in \mathbb{Z}} \Theta_{1m}^p \right)^{1/p} \leq \sum_{m \in \mathbb{Z}} \Theta_{1m} \lesssim \ell(D) \int_{(3D)c} |z_D - y|^{-n-1} \, d\mathcal{H}_1^n(y) \lesssim 1.$$

The case of $\Theta_{2m}$ is more delicate. Since $\Gamma$ is a Lipschitz graph, there exists an integer $M > 10$ depending only on $\text{Lip}(A)$ such that any $x \in \Gamma \cap D$ satisfies $|x - z_D| < 2^M \ell(D)$. Without loss of generality, we can assume that there exists $m_0 \in \mathbb{Z}$ such that $\epsilon_{m_0} = 2^{M+2}\ell(D)$, just by adding the term $2^{M+2}\ell(D)$ to the fixed sequence $\{\epsilon_m\}_{m \in \mathbb{Z}}$.

We set $J_0 := \{m \in \mathbb{Z} : \epsilon_m \leq 2^{M+2}\ell(D)\} = \{m \in \mathbb{Z} : m \geq m_0\}$ and, for $j > M + 2$,

$$J_j^1 := \{m \in \mathbb{Z} : 2^{j-1}\ell(D) \leq \epsilon_{m+1} < \epsilon_m \leq 2^j\ell(D) \text{ and } \epsilon_m - \epsilon_{m+1} \geq 2^M \ell(D)\},$$

$$J_j^2 := \{m \in \mathbb{Z} : 2^{j-1}\ell(D) \leq \epsilon_{m+1} < \epsilon_m \leq 2^j\ell(D) \text{ and } \epsilon_m - \epsilon_{m+1} < 2^M \ell(D)\},$$

$$J_j^3 := \{m \in \mathbb{Z} : 2^{j-1}\ell(D) \leq \epsilon_{m+1} < 2^j\ell(D) < \epsilon_m\}.$$

Then $Z = J_0 \cup \left( \bigcup_{j > M+2} (J_j^1 \cup J_j^2 \cup J_j^3) \right)$. For the case of $m \in J_0$, we have the easy estimate

$$\left( \sum_{m \in J_0} \Theta_{2m}^p \right)^{1/p} \lesssim \sum_{m \in J_0} \int_{(3D)c} \left( \chi_{\epsilon_{m+1}}(x - y) + \chi_{\epsilon_{m}}(z_D - y) \right) \ell(D)^{-n} \, d\mathcal{H}_1^n(y)$$

$$\leq \int_{|x - y| < 2^{M+2}\ell(D)} \frac{d\mathcal{H}_1^n(y)}{\ell(D)^n} + \int_{|z_D - y| \leq 2^{M+2}\ell(D)} \frac{d\mathcal{H}_1^n(y)}{\ell(D)^n} \lesssim 1.$$

Assume that $m \in J_j^1$. Notice that $\text{supp}(\chi_{\epsilon_{m+1}}(x - \cdot) - \chi_{\epsilon_{m}}(z_D - \cdot)) \subset A_m(x, z_D)$, where $A_m(x, z_D)$ denotes the symmetric difference between $A(x, \epsilon_{m+1}, \epsilon_m)$ and $A(z_D, \epsilon_{m+1}, \epsilon_m)$. Notice also that, since $m \in J_j^1$ and $x \in D \cap \Gamma$, the set $A_m(x, z_D)$ is contained in the union of
annuli $A_1 := A(x, \epsilon_{m+1} - 2^M \ell(D), \epsilon_{m+1} + 2^M \ell(D))$ and $A_2 := A(x, \epsilon_{m} - 2^M \ell(D), \epsilon_{m} + 2^M \ell(D))$. For $z \in \Gamma$ and $0 < a \leq b$, we have $\mathcal{H}_r^n(A(z, a, b)) \lesssim (b-a)b^{n-1}$ by Lemma 4.1.2. Hence, since $m \in J^1_j$,

$$\mathcal{H}_r^n(\{y \in \mathbb{R}^d : |\chi_{\epsilon_{m+1}}(x - y) - \chi_{\epsilon_{m+1}}(z_D - y)| \neq 0\}) \leq \mathcal{H}_r^n(A_1 \cup A_2) \lesssim 2^{M+1}\ell(D)(\epsilon_m + 2^M \ell(D))^{n-1} + 2^{M+1}\ell(D)(\epsilon_{m+1} + 2^M \ell(D))^{n-1} \lesssim 2^{j(n-1)}\ell(D)^n.$$  

(4.2.1)

Using that $|K(z_D - y)| \lesssim (2^j\ell(D))^{-n}$ for all $y \in A_m(x, z_D) \cap (3D)^c$, we get

$$\Theta_{2m} \lesssim (2^j\ell(D))^{-n}2^{j(n-1)}\ell(D)^n = 2^{-j}$$

and, since $J^1_j$ contains at most $2^{-j-M-1}$ indices and $\rho > 2$, we have $\sum_{m \in J^1_j} \Theta_{2m} \lesssim 2^{-j}$.

Assume now that $m \in J^2_j$. Then, using Lemma 4.1.2, we obtain

$$\mathcal{H}_r^n(\{y \in \mathbb{R}^d : |\chi_{\epsilon_{m+1}}(x - y) - \chi_{\epsilon_{m+1}}(z_D - y)| \neq 0\}) \leq \mathcal{H}_r^n(\{y \in \mathbb{R}^d : \chi_{\epsilon_{m+1}}(x - y) = 1\}) + \mathcal{H}_r^n(\{y \in \mathbb{R}^d : \chi_{\epsilon_{m+1}}(z_D - y) = 1\}) \lesssim (\epsilon_m - \epsilon_{m+1})\epsilon_{m-1}^{-n},$$

and, as above, $|K(z_D - y)| \lesssim (2^j\ell(D))^{-n}$ for all $y \in A_m(x, z_D) \cap (3D)^c$. Since $m \in J^2_j$,

$$\Theta_{2m} \lesssim (2^j\ell(D))^{-n}(\epsilon_m - \epsilon_{m+1})\epsilon_{m-1}^{-n} \lesssim (2^j\ell(D))^{-n}(\epsilon_m - \epsilon_{m+1})(2^M\ell(D))^{n-1}(2^j\ell(D))^{n-1} \lesssim 2^{-j}\ell(D)^{-1}(\epsilon_m - \epsilon_{m+1})$$

and then, since $\rho > 2$ and $j > M + 2 > 12$,

$$\sum_{m \in J^2_j} \Theta_{2m} \lesssim 2^{-j}\sum_{m \in J^2_j} \frac{\epsilon_m - \epsilon_{m+1}}{\ell(D)} \lesssim 2^{-j}\rho 2^{-j-1} \approx 2^{-j(\rho-1)} \leq 2^{-j}.$$

Finally, assume that $m \in J^3_j$. Obviously, the set $J^3_j$ contains at most one term. If $\epsilon_m - \epsilon_{m+1} < 2^M\ell(D)$, arguing as in the case $m \in J^2_j$, we have

$$\mathcal{H}_r^n(\{y \in \mathbb{R}^d : |\chi_{\epsilon_{m+1}}(x - y) - \chi_{\epsilon_{m+1}}(z_D - y)| \neq 0\}) \lesssim (\epsilon_m - \epsilon_{m+1})\epsilon_{m-1}^{-n} \lesssim 2^M\ell(D)(2^j\ell(D))^2 + 2^M\ell(D))^{n-1} \lesssim 2^{j(n-1)}\ell(D)^n,$$

and then $\Theta_{2m} \lesssim 2^{j(n-1)}\ell(D)(2^j\ell(D))^{-n} \lesssim 2^{-j}$. On the contrary, if $\epsilon_m - \epsilon_{m+1} \geq 2^M\ell(D)$, arguing as in the case $m \in J^1_j$, we have $\text{supp}(\chi_{\epsilon_{m+1}}(x - \cdot) - \chi_{\epsilon_{m+1}}(z_D - \cdot)) \subset A_m(x, \epsilon_D) \subset A_1 \cup A_2$. Similarly to (4.2.1), we have

$$\mathcal{H}_r^n(A_1) \lesssim 2^{M+1}\ell(D)(\epsilon_{m+1} + 2^M\ell(D))^{n-1} \lesssim \epsilon_{m+1}^{-n-1}\ell(D) \lesssim 2^{j(n-1)}\ell(D)^n,$$

and $|K(z_D - y)| \lesssim (2^j\ell(D))^{-n}$ for all $y \in A_1 \cap (3D)^c$. If we denote by $j(\epsilon_m)$ the positive integer such that $2^{j(\epsilon_m)-1}\ell(D) \leq \epsilon_m \leq 2^{j(\epsilon_m)}\ell(D)$ (obviously, $j(\epsilon_m) > j$), we have $\mathcal{H}_r^n(A_2) \lesssim 2^{j(n-1)}\ell(D)^n.$
\(\epsilon_m^{n-1}\ell(D) \leq 2^{j(\ell_m)(n-1)}\ell(D)^n\), and \(|K(z_D - y)| \lesssim (2^{j(\ell_m)}\ell(D))^{-n}\) for all \(y \in A_2 \cap (3D)^c\). Hence, \(\Theta_{2m} \lesssim 2^{j(n-1)\ell(D)^n}2^{j(\ell_m)(n-1)\ell(D)^n}2^{j(\ell_m)\ell(D)^n} \lesssim 2^{-j} + 2^{-j(\ell_m)} \lesssim 2^{-j}\).

Therefore, since \(J_j^3\) contains at most one term, \(\sum_{m \in J_j^3} \Theta_{2m}^p \lesssim 2^{-jn} \lesssim 2^{-j}\).

We put all these estimates together and conclude that

\[
|\left(\mathcal{V}_p \circ \mathcal{T}^{H^m_\Gamma}_x\right) f_2(x) - c | \lesssim \|f\|_{L^\infty(H^m_\Gamma)} \left( \sum_{m \in \mathbb{Z}} (\Theta_{1m} + \Theta_{2m})^p \right)^{1/p},
\]

\[
\lesssim \|f\|_{L^\infty(H^m_\Gamma)} \left( \sum_{m \in \mathbb{Z}} \Theta_{1m}^p \right)^{1/p} + \|f\|_{L^\infty(H^m_\Gamma)} \left( \sum_{m \in J_0} \Theta_{2m}^p \right)^{1/p} + \|f\|_{L^\infty(H^m_\Gamma)} \left( \sum_{j > M+2} \left( \sum_{m \in J_j^3} \Theta_{2m}^p + \sum_{m \in J_j^3} \Theta_{2m}^p + \sum_{m \in J_j^3} \Theta_{2m}^p \right) \right)^{1/p}
\]

\[
\lesssim \|f\|_{L^\infty(H^m_\Gamma)} \left( 1 + 1 + \left( \sum_{j > 12} 2^{-j}\right)^{1/p} \right) \lesssim \|f\|_{L^\infty(H^m_\Gamma)},
\]

and so the boundedness of \(\mathcal{V}_p \circ \mathcal{T}^{H^m_\Gamma}_x : L^\infty(H^m_\Gamma) \to BMO(H^m_\Gamma)\) follows, as in Subsection 3.6.2.

### 4.2.3 The operator \(\mathcal{V}_p \circ \mathcal{T}^{H^m_\Gamma}_x : L^p(H^m_\Gamma) \to L^p(H^m_\Gamma)\) is bounded for \(1 < p < \infty\)

We deduce the \(L^p\) boundedness of the positive sublinear operator \(\mathcal{V}_p \circ \mathcal{T}^{H^m_\Gamma}_x\) by interpolation between the pairs \((L^1(H^m_\Gamma), L^1(H^m_\Gamma))\) and \((L^2(H^m_\Gamma), L^2(H^m_\Gamma))\) for \(1 < p < 2\), and between \((L^2(H^m_\Gamma), L^2(H^m_\Gamma))\) and \((L^\infty(H^m_\Gamma), BMO(H^m_\Gamma))\) for \(2 < p < \infty\).

Let us remark that, in the latter case, the classical interpolation theorem (see [Du, Theorem 6.8], for instance) would require the operator \(\mathcal{V}_p \circ \mathcal{T}^{H^m_\Gamma}_x\) to be linear. Clearly, this fails in our case. However, an easy modification of the arguments in [Du] using Lemma 3.6.2 shows that the interpolation theorem between \((L^2, L^2)\) and \((L^\infty, BMO)\) is also valid for positive sublinear operators.

**Remark 4.2.2.** From Remark 4.1.7, we know that the operators \(\mathcal{V}_p \circ \mathcal{T}^{H^m_\Gamma}_x\) and \(\mathcal{V}_p \circ \mathcal{T}^{H^m_\Gamma}_x\) are also bounded in \(L^2(H^m_\Gamma)\). The endpoint estimates and the interpolation theorem can also be obtained for these operators by very similar arguments: for the family of truncations \(\tilde{\chi}\) one argues as for \(\chi\) (but now Lemma 4.1.2 is not necessary), and for \(\varphi\) one uses the regularity of the functions \(\varphi\), as we pointed out in Remark 4.1.7 (for example, the proof of the boundedness of \(\mathcal{V}_p \circ \mathcal{T}^{H^m_\Gamma}_x : L^\infty(H^m_\Gamma) \to BMO(H^m_\Gamma)\) is analogous to the one in Subsection 3.6.2). The same holds for the operators \(\mathcal{O} \circ \mathcal{T}^{H^m_\Gamma}_x\) and \(\mathcal{O} \circ \mathcal{T}^{H^m_\Gamma}_x\).

For the case of \(\mathcal{V}_p \circ \mathcal{T}^{H^m_\Gamma}_x\) and \(\mathcal{O} \circ \mathcal{T}^{H^m_\Gamma}_x\), the weak \(L^1\) estimate can be obtained similarly to the case of the family of truncations \(\varphi\), that is, by adapting Theorem B of [CJRW2] to our specific setting.
We want to emphasize that the assumption $\text{Lip}(A) < 1$ is not necessary when we deal with any of the families of truncations $\tilde{\chi}$, $\varphi$ or $\tilde{\varphi}$ (see the comment that follows Lemma 4.1.2). Therefore, Main Theorem 3.0.1 is finally proven.
Chapter 5

Variation and oscillation for Riesz transforms and uniform rectifiability

This chapter is a natural continuation of Chapters 3 and 4. As before, the kernels $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ that we consider here satisfy

$$|K(x)| \leq \frac{C}{|x|^n}, \quad |\partial_{x_i} K(x)| \leq \frac{C}{|x|^{n+1}} \quad \text{and} \quad |\partial_{x_i} \partial_{x_j} K(x)| \leq \frac{C}{|x|^{n+2}}, \quad (5.0.1)$$

for all $1 \leq i, j \leq d$ and $x = (x^1, \ldots, x^d) \in \mathbb{R}^d \setminus \{0\}$, where $0 < n < d$ is some integer and $C > 0$ is some constant; and moreover $K(-x) = -K(x)$ for all $x \neq 0$ (i.e. $K$ is odd). First of all, let us recall some definitions.

**Definition 5.0.3.** Let $\chi_\mathbb{R} := \chi_{[1, \infty)}$ and let $\varphi_\mathbb{R} : [0, +\infty) \to [0, +\infty)$ be a non decreasing $C^2$ function with $\chi_{[4, \infty)} \leq \varphi_\mathbb{R} \leq \chi_{[1/4, \infty)}$. Suppose moreover that $|\varphi_\mathbb{R}'|$ is bounded below away from zero in $[1/3, 3]$, i.e., $\chi_{[1/3, 3]} \leq C|\varphi_\mathbb{R}'|$. Given $x \in \mathbb{R}^d$ and $0 < \epsilon \leq \delta$, we set $\chi_\epsilon(x) := \chi_\mathbb{R}(|x|/\epsilon)$, $\chi_\delta(x) := \chi_\epsilon(x) - \chi_\delta(x)$, $\varphi_\epsilon(x) = \varphi_\mathbb{R}(|x|^2/\epsilon^2)$, and $\varphi_\delta(x) = \varphi_\epsilon(x) - \varphi_\delta(x)$.

Given a finite Borel measure $\mu$, we define

$$T_\epsilon \mu(x) := (K \chi_\epsilon * \mu)(x) = \int \chi_\epsilon(x-y)K(x-y)\,d\mu(y).$$

$$T_{\varphi_\epsilon} \mu(x) := (K \varphi_\epsilon * \mu)(x) = \int \varphi_\epsilon(x-y)K(x-y)\,d\mu(y).$$

Finally, we set $T_\mu := \{T_\epsilon \mu\}_{\epsilon > 0}$ and $T_{\varphi_\epsilon} \mu := \{T_{\varphi_\epsilon} \mu\}_{\epsilon > 0}$, and for $f \in L^1(\mu)$, we set $T_\epsilon^\mu f := T_\epsilon(f \mu)$, $T_{\varphi_\epsilon}^\mu f := T_{\varphi_\epsilon}(f \mu)$, $T_\mu^f := \{T_\epsilon^f \mu\}_{\epsilon > 0}$, and $T_{\varphi_\epsilon}^f := \{T_{\varphi_\epsilon}^f \mu\}_{\epsilon > 0}$.

**Remark 5.0.4.** In the definition, the choice of $[4, \infty)$, $[1/4, \infty)$, and $[1/3, 3]$ is not specially relevant, it is just for definiteness. One can replace the numbers 3 and 4 in the definition of $\varphi_\mathbb{R}$ by any other two positive numbers $0 < a < b$, and all the proofs in the paper remain almost the same.
Definition 5.0.5 ($\rho$-variation and oscillation). Let $F := \{F_r\}_{r > 0}$ be a family of functions defined on $\mathbb{R}^d$. Given $\rho > 0$, the $\rho$-variation of $F$ at $x \in \mathbb{R}^d$ is defined by

$$V_\rho(F)(x) := \sup_{\{\epsilon_m\}} \left( \sum_{m \in \mathbb{Z}} |F_{\epsilon_{m+1}}(x) - F_{\epsilon_m}(x)|^\rho \right)^{1/\rho},$$

where the pointwise supremum is taken over all decreasing sequences $\{\epsilon_m\}_{m \in \mathbb{Z}} \subset (0, \infty)$. Fix a decreasing sequence $\{r_m\}_{m \in \mathbb{Z}} \subset (0, \infty)$. The oscillation of $F$ at $x \in \mathbb{R}^d$ is defined by

$$O(F)(x) := \sup_{\{\epsilon_m\}, \{\delta_m\}} \left( \sum_{m \in \mathbb{Z}} |F_{\epsilon_m}(x) - F_{\delta_m}(x)|^2 \right)^{1/2},$$

where the pointwise supremum is taken over all sequences $\{\epsilon_m\}_{m \in \mathbb{Z}}$ and $\{\delta_m\}_{m \in \mathbb{Z}}$ such that $r_{m+1} \leq \epsilon_m \leq \delta_m \leq r_m$ for all $m \in \mathbb{Z}$.

In this chapter we continue the study developed in Chapters 3 and 4 about the $\rho$-variation and oscillation of the families $T^\mu f$ and $T_\rho^s f$. That is, given a Borel measure $\mu$ and $f \in L^1(\mu)$ we will deal with $(V_\rho \circ T^\mu f)(x) = (V_\rho \circ T)(f \mu)(x) := V_\rho(T^\mu f)(x)$ and $(O \circ T^\mu f)(x) = (O \circ T)(f \mu)(x) := O(T^\mu f)(x)$, and analogously for $T_\rho^s f$. Mainly, our interest is focused in the case where $K$ is the (scalar components of) the Riesz kernel.

G. David and S. Semmes asked more than twenty years ago the still open question that follows:

Question 5.0.6. Is it true that an $n$-dimensional AD regular measure $\mu$ is uniformly $n$-rectifiable if and only if $R^\mu_x$ is bounded in $L^2(\mu)$?

Some comments are in order. In [DS1], G. David and S. Semmes proved the “only if” implication of question above. Moreover, they gave a positive answer if one replaces, in Question 5.0.6, the $L^2$ boundedness of $R^\mu_x$ by the $L^2$ boundedness of $T_\rho^s f$ for a wide class of odd kernels $K$. The “if” implication was proved by P. Mattila, M. Melnikov and J. Verdera in [MMV] only for the case of the Cauchy transform, i.e., $n = 1$ and $d = 2$. Later on, in [Le] G. David and J. C. Léger proved that the $L^2$ boundedness $C^\mu_\rho$ implies that $\mu$ is rectifiable, i.e., they obtained the corresponding “if” implication without the AD regularity assumption (with $n = 1$ and $d = 2$).

When $\mu$ is the $n$-dimensional Hausdorff measure on a set $E \subset \mathbb{R}^d$ such that $\mu(E) < \infty$, the rectifiability of $\mu$ is also related with the existence of the principal value of the Riesz transforms of $\mu$, that is, the existence of $R^\mu 1(x) = \lim_{r \to 0} R^\mu r 1(x)$ for $\mu$-a.e. $x \in E$. In [MPr], P. Mattila and D. Preiss proved that, under the additional assumption that $\liminf_{r \to 0} r^{-n} \mu(B(x, r)) > 0$ for $\mu$-a.e. $x \in E$, the rectifiability of $E$ is equivalent to the existence of $R^\mu 1(x)$ $\mu$-a.e. $x \in E$. Later on, in [To10] X. Tolsa removed the assumption on
the lower density of $\mu$, i.e., he proved that $E$ is rectifiable if and only if the principal values $R^\mu 1$ exists $\mu$ almost everywhere. Let us mention that, for the case $n = 1$ and $d = 2$ (that is, for the Cauchy transform), the same results where obtained in [Ma2] (with some density assumptions) and in [To3] (by using the notion of curvature of measures).

The purpose of this chapter is twofold. The first and most important one (for us) is the following theorem, which is intimately related with Question 5.0.6. Recall that $R^\mu = \{R^\mu_\epsilon\}_{\epsilon > 0}$.

**Theorem 5.0.7.** Let $0 < n < d$ be integers. Let $\mu$ be an $n$-dimensional AD regular Borel measure on $\mathbb{R}^d$. The following are equivalent:

(a) $\mu$ is uniformly $n$-rectifiable.

(b) For any $K$ like in (5.0.1) and any $\rho > 2$, the operator $V_\rho \circ T^\mu_\varphi$ is bounded in $L^p(\mu)$ for all $1 < p < \infty$, and from $L^1(\mu)$ into $L^{1,\infty}(\mu)$.

(c) For any $K$ like in (5.0.1) and any $\rho > 2$, the operator $V_\rho \circ T^\mu_\varphi$ is bounded in $L^2(\mu)$.

(d) For some $\rho > 0$, the operator $V_\rho \circ R^\mu$ is bounded in $L^2(\mu)$.

(e) For $K(x) = x/|x|^{n+1}$ and for some $\rho > 0$, the operator $V_\rho \circ T^\mu_\varphi$ is bounded in $L^2(\mu)$.

**Remark 5.0.8.** Let $\{r_m\}_{m \in \mathbb{Z}} \subset (0, \infty)$ be a fixed decreasing sequence defining $O$. Then, the implications (a) $\Rightarrow$ (b), . . ., (e) in the theorem above still hold if one replaces $V_\rho$ by $O$. If there exists $C > 0$ such that $C^{-1}r_m \leq r_m - r_{m+1} \leq Cr_m$ for all $m \in \mathbb{Z}$, then the implications (b), . . ., (e) $\Rightarrow$ (a) also hold (so Theorem 5.0.7 remains true replacing $V_\rho$ by $O$), but we do not know if they still hold without this additional assumption (see Remark 5.4.9).

**Remark 5.0.9.** Despite (d) and (e) of Theorem 5.0.7 being stated stated in terms of some $\rho > 0$, in general the $\rho$-variation fails to be bounded for $\rho \leq 2$ (we will not prove this fact). Furthermore, if it is bounded for some $\rho > 0$, then it is also bounded for all $\tilde{\rho} \geq \rho$. Thus, in the practice we will always have $\rho > 2$, and one can replace “for some $\rho > 0$” by “for some $\rho > 2$” in (d) and (e) of Theorem 5.0.7.

Notice that, by Theorem 5.0.7 and Remark 5.0.8, the operators $V_\rho \circ R^\mu$ and $O \circ R^\mu$ completely characterize the $n$-AD regular measures $\mu$ which are uniformly rectifiable. The same happens with $V_\rho \circ T^\mu_\varphi$ and $O \circ T^\mu_\varphi$ for $K(x) = x/|x|^{n+1}$. Thus, as a direct consequence of the theorem, we obtain the following corollary, which is the main motivation of this chapter because it might be considered as a partial answer to Question 5.0.6.

**Corollary 5.0.10.** Fix $\rho > 2$. An $n$-dimensional AD regular Borel measure $\mu$ on $\mathbb{R}^d$ is uniformly rectifiable if and only if the operator $V_\rho \circ R^\mu$ is bounded in $L^2(\mu)$.
The proof of Theorem 5.0.7 is the following: the implication \((a) \implies (b)\) is a straightforward application of Corollary 5.2.2 below, the implication \((a) \implies (c)\) is given by Theorem 5.3.1, the implications \((b) \implies (e)\) and \((c) \implies (d)\) are obvious, and the implications \((d) \implies (e) \implies (a)\) are given by (the proof of) Theorem 5.4.8.

Let us mention that, in Theorem 5.0.7, the implication \((a) \implies (b)\) will be proved using a good \(\lambda\) inequality, and at some point we will need the estimate \(|(\mathcal{V}_\rho \circ T^\mu) f(x) - (\mathcal{V}_\rho \circ T^\mu) f(z)| \leq M^\mu f(x)\), where \(x, z \in B\) for some fixed ball \(B \subset \mathbb{R}^d\), \(f\) is a function supported out of \(B\), and \(M^\mu\) denotes the Hardy-Littlewood maximal operator with respect to \(\mu\). This estimate will be obtained using the regularity of \(\varphi\) (see (5.2.4) and (5.2.5)). However, we have not been able to show this estimate replacing \(\mathcal{V}_\rho \circ T^\mu\) by \(\mathcal{V}_\rho \circ T^\mu\) (it might be false), and this is the main obstacle for replacing \(L^2(\mu)\) by \(L^p(\mu)\) \((1 < p < \infty)\) and \(L^1(\mu) \to L^{1,\infty}(\mu)\) in \((c)\) of Theorem 5.0.7.

The second purpose of this chapter concerns variational endpoint estimates for singular integrals on Lipschitz graphs, which is also a natural continuation of the results obtained in Chapters 3 and 4. Recall that by an \(n\)-dimensional Lipschitz graph \(\Gamma \subset \mathbb{R}^d\) we mean any translation and rotation of a set of the type \(\{x \in \mathbb{R}^d : x = (y, A(y)), y \in \mathbb{R}^n\}\), where \(A: \mathbb{R}^n \to \mathbb{R}^{d-n}\) is some Lipschitz function with Lipschitz constant \(\text{Lip}(A)\). We say that \(\text{Lip}(A)\) is the slope of \(\Gamma\). Main Theorem 3.0.1 of Chapter 3 contains the following result:

**Theorem 5.0.11.** Let \(\rho > 2\). Let \(\Gamma \subset \mathbb{R}^d\) be an \(n\)-dimensional Lipschitz graph and set \(\mu := \mathcal{H}^d_\Gamma\). The operators \(\mathcal{V}_\rho \circ T^\mu\) and \(\mathcal{O} \circ T^\mu\) are bounded

- in \(L^p(\mu)\) for \(1 < p < \infty\),
- from \(L^1(\mu)\) to \(L^{1,\infty}(\mu)\), and
- from \(L^\infty(\mu)\) to \(\text{BMO}(\mu)\).

The same holds for \(\mathcal{V}_\rho \circ T^\mu\) and \(\mathcal{O} \circ T^\mu\) if the slope of \(\Gamma\) is strictly less than 1. The norms of these operators are bounded by some constants depending only on \(n, d, K, p, \rho, \varphi_\mathbb{R}\), and the maximum slope of \(\Gamma\). In particular, the norms of \(\mathcal{O} \circ T^\mu\) and \(\mathcal{O} \circ T^\mu\) in the cases above are bounded independently of the sequence that defines \(\mathcal{O}\).

The following theorem, which is the second purpose of this chapter, improves the first endpoint estimate of Theorem 5.0.11. Denote by \(M(\mathbb{R}^d)\) the space of finite complex Radon measures on \(\mathbb{R}^d\).

**Theorem 5.0.12.** Let \(\rho > 2\) and let \(\mu\) be the \(n\)-dimensional Hausdorff measure restricted to an \(n\)-dimensional Lipschitz graph \(\Gamma \subset \mathbb{R}^d\).

(a) The operator \(\mathcal{V}_\rho \circ T^\mu\) is bounded from \(M(\mathbb{R}^d)\) to \(L^{1,\infty}(\mu)\).

(b) If the slope of \(\Gamma\) is strictly less than 1, \(\mathcal{V}_\rho \circ T^\mu\) is a bounded operator from \(M(\mathbb{R}^d)\) to \(L^{1,\infty}(\mu)\).
5.1. Boundedness of $V_\rho \circ T_\varphi$ from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mathcal{H}_\Gamma^n)$

In particular, $V_\rho \circ T_\mu$ and $V_\rho \circ T^\mu_\varphi$ are of weak type $(1,1)$. In both cases, the bound of the norm of this operator only depends on $n$, $d$, $K$, $\rho$, $\varphi_\mathbb{R}$ (in the first case), and the slope of $\Gamma$.

**Remark 5.0.13.** The theorem above also holds if one replaces $V_\rho$ by $O$. Moreover, the norm of $O \circ T^\mu_\varphi$ in the cases above is bounded independently of the sequence that defines $O$.

The plan of this chapter is the following: in Section 5.1 we prove Theorem 5.0.12(a). In Section 5.2 we prove Theorem 5.2.1 and Corollary 5.2.2, which gives the implication $(a) \implies (b)$ of Theorem 5.0.7, and which is proven using Theorem 5.0.12(a). In Section 5.3 we prove Theorem 5.3.1, which yields $(a) \implies (c)$ of Theorem 5.0.7, and in Section 5.4 we prove Theorem 5.4.8, which gives the implications $(d) \implies (e) \implies (a)$, and finishes the proof of Theorem 5.0.7. Finally, in Section 5.5 we prove Theorem 5.0.12(b).

**Remark 5.0.14.** For proving Theorem 5.0.7, one uses part $(a)$ of Theorem 5.0.12, but part $(b)$ is not necessary. More precisely, one only uses the $L^2$ boundedness of $V_\rho \circ T^\mu_\varphi$ on Lipschitz graphs, which is proved in Chapter 4. However, when $\mu$ is the Hausdorff measure restricted to a Lipschitz graph, the $L^2$ boundedness of $V_\rho \circ T^\mu_\varphi$ can be obtained with very similar techniques to the ones in the proof of the $L^2(\mu)$ boundedness of $V_\rho \circ T^\mu_\varphi$ in Chapter 3. That is, if $\mu$ is as in Theorem 3.1.7, one splits $V_\rho \circ T^\mu_\varphi$ into short and long variation. Then, one deals with the short variation as for the case of the operator $V_\rho \circ T^\mu_\tilde{\varphi}$ (i.e., when we truncate in the space of coordinates relative to the graph), and one deals with the long variation by comparing $V_\rho \circ T^\mu_\varphi$ with $V_\rho \circ T^\mu_\tilde{\varphi}$ and by estimating the error terms by means of the $\alpha$ coefficients. We will not give the details of this alternative proof of Theorem 5.0.7 which skips Chapter 4 because the techniques involved are very similar to the ones already used in this book.

Theorems 5.0.7 and 5.0.12 are stated in terms of $V_\rho$, but they also hold for $O$, as it is said in the remarks below them. However, we will only give the proof of these results for $V_\rho$, because the case of $O$ follows by very similar arguments and computations (taking into account Remark 5.0.8). Furthermore, the same applies to the rest of the paper. The details are left for the reader.

### 5.1 Boundedness of $V_\rho \circ T_\varphi$ from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mathcal{H}_\Gamma^n)$

The proof of Theorem 5.0.12 is a modification of the proof of [CJRW2, Theorem B] using a Calderón-Zygmund decomposition of measures in $M(\mathbb{R}^d)$ with respect to $\mathcal{H}_\Gamma^n$.

#### 5.1.1 Calderón-Zygmund decomposition for general measures

Given $\nu \in M(\mathbb{R}^d)$, $a > 1$ and $b > a^n$, we say that a cube $Q$ is $(a,b)$-$|\nu|$-doubling if $\nu(aQ) \leq b\nu(Q)$, where $aQ$ is the cube concentric with $Q$ with side length $a\ell(Q)$. For definiteness, if $a$
and $b$ are not specified, by a doubling cube we mean a $(2, 2^{d+1})$-\(|\nu|\)-doubling cube.

The following two lemmas are already known (see [To5], [To4], or [To12] for example), but since they are essential in this chapter, we give their proof for completeness.

**Lemma 5.1.1.** Let $b > a^d$. If $\nu$ is a Radon measure in $\mathbb{R}^d$, then for $\nu$-a.e. $x \in \mathbb{R}^d$ there exists a sequence of $(a, b)$-\(|\nu|\)-doubling cubes $\{Q_k\}_k$ centered at $x$ with $\ell(Q_k) \to 0$ as $k \to \infty$.

**Proof.** Let $Z \subseteq \mathbb{R}^d$ be the set of points $x$ such that there does not exist a sequence of $(a, b)$-\(|\nu|\)-doubling cubes $\{Q_k\}_{k \geq 0}$ centered at $x$ with side length decreasing to 0; and let $Z_j \subseteq \mathbb{R}^d$ be the set of points $x$ such that there does not exist any $(a, b)$-\(|\nu|\)-doubling cube $Q$ centered at $x$ with $\ell(Q) \leq 2^{-j}$. Clearly, $Z = \bigcup_{j \geq 0} Z_j$. Thus, proving the lemma is equivalent to showing that $\nu(Z_j) = 0$ for every $j \geq 0$.

Let $Q_0$ be a fixed cube with side length $2^{-j}$ and let $k \geq 1$ be some integer. For each $z \in Q_0 \cap Z_j$, let $Q_z$ be a cube centered at $z$ with side length $a^{-k}\ell(Q_0)$. Since the cubes $a^k Q_z$ are not $(a, b)$-\(|\nu|\)-doubling for $h = 0, \ldots, k - 1$ and $a^k Q_z \subset 2Q_0$, we have

$$\nu(Q_z) \leq b^{-1}(aQ_z) \leq \cdots \leq b^{-k}(a^k Q_z) \leq b^{-k}(2Q_0). \quad (5.1.1)$$

By Besicovitch’s theorem, there exists a subfamily $\{z_m\}_m \subseteq Q_0 \cap Z_j$ such that $Q_0 \cap Z_j \subset \bigcup_m Q_{z_m}$ and moreover $\sum_m \chi_{Q_{z_m}} \leq P_d$. This is a finite family and the number $N$ of points $z_m$ can be easily bounded above as follows: if $\mathcal{L}$ stands for the Lebesgue measure on $\mathbb{R}^d$,

$$N(a^{-k}\ell(Q_0))^d = \sum_{m=1}^N \mathcal{L}(Q_{z_m}) \leq P_d \mathcal{L}(2Q_0) = P_d(2\ell(Q_0))^d.$$

Thus, $N \leq P_d 2^d a^{kd}$. As a consequence, since $\{Q_{z_m}\}_{1 \leq m \leq N}$ covers $Q_0 \cap Z_j$, by (5.1.1),

$$\nu(Q_0 \cap Z_j) \leq \sum_{m=1}^N \nu(Q_z) \leq Nb^{-k}\nu(2Q_0) \leq P_d 2^d a^{kd} b^{-k}\nu(2Q_0).$$

Since $b > a^d$, the right hand side tends to 0 as $k \to \infty$. Therefore $\nu(Q_0 \cap Z_j) = 0$, and since the cube $Q_0$ is arbitrary, we are done. \hfill \□

**Lemma 5.1.2 (Calderón-Zygmund decomposition).** Assume that $\mu := \mathcal{H}^n_{\Gamma \cap B}$, where $\Gamma$ is an $n$-dimensional Lipschits graph and $B \subset \mathbb{R}^d$ is some fixed ball. For every $\nu \in M(\mathbb{R}^d)$ with compact support and every $\lambda > 2^{d+1}\nu/\|\mu\|$, we have:

(a) There exists a finite or countable collection of almost disjoint cubes $\{Q_j\}_j$ (that is, $\sum_j \chi_{Q_j} \leq C$) and a function $f \in L^1(\mu)$ such that

$$|\nu|(Q_j) > 2^{-d-1}\lambda \mu(2Q_j), \quad (5.1.2)$$

$$|\nu|\eta Q_j) \leq 2^{-d-1}\lambda \mu(2\eta Q_j) \quad \text{for } \eta > 2, \quad (5.1.3)$$

$$\nu = f \mu \text{ in } \mathbb{R}^d \setminus \Omega \text{ with } |f| \leq \lambda \mu\text{-a.e.}, \text{ where } \Omega = \bigcup_j Q_j. \quad (5.1.4)$$
(b) For each \( j \), let \( R_j := 6Q_j \) and denote \( w_j := \chi_{Q_j}(\sum_k \chi_{Q_k})^{-1} \). Then, there exists a family of functions \( \{b_j\}_j \) with \( \text{supp} b_j \subset R_j \) and with constant sign satisfying

\[
\int b_j \, d\mu = \int w_j \, d\nu, \quad (5.1.5)
\]

\[
\|b_j\|_{L^{\infty}(\mu)} \mu(R_j) \leq C|\nu|(Q_j), \quad \text{and} \quad (5.1.6)
\]

\[
\sum_j |b_j| \leq C_0 \lambda \quad (\text{where } C_0 \text{ is some absolute constant}). \quad (5.1.7)
\]

**Proof of Lemma 5.1.2(a).** Let \( H \) be the set of those points from \( \text{supp} \mu \cup \text{supp} \nu \) such that there exists some cube \( Q \) centered at \( x \) satisfying \(|\nu|(Q) > 2^{-d-1}\lambda \mu(2Q)\). For each \( x \in H \), let \( Q_x \) be a cube centered at \( x \) such that the preceding inequality holds for \( Q_x \) but fails for the cubes \( Q \) centered at \( x \) with \( \ell(Q) > 2\ell(Q_x) \). Notice that the condition \( \lambda > 2^{d+1} \|\nu\|/\|\mu\| \) guaranties the existence of \( Q_x \).

Since \( H \) is bounded (because \( \mu \) and \( \nu \) are compactly supported), we can apply Besicovitch’s covering theorem to get a finite or countable almost disjoint subfamily of cubes \( \{Q_j\}_j \subset \{Q_x\}_{x \in H} \) which cover \( H \) and satisfy (5.1.2) and (5.1.3) by construction.

To prove (5.1.4), denote by \( Z \) be the set of points \( y \in \text{supp} \nu \) such there does not exist a sequence of \( (2, 2^{d+1})\)-\( |\nu| \)-doubling cubes centered at \( y \) with side length tending to 0, so that \(|\nu|(Z) = 0\), by Lemma 5.1.1. By the definitions of \( H \) and \( Z \), for every \( x \in \text{supp} \nu \setminus (H \cup Z) \), there exists a sequence of \( (2, 2^{d+1})\)-\( |\nu| \)-doubling cubes \( P_k \) centered at \( x \), with \( \ell(P_k) \to 0 \), such that \(|\nu|(P_k) \leq 2^{-d-1}\lambda \mu(2P_k) \), and thus \(|\nu|(2P_k) \leq 2^{d+1}|\nu|(P_k) \leq \lambda \mu(2P_k) \). This implies that \( \chi_{\mathbb{R}^d \setminus (H \cup Z)} \nu = \chi_{\mathbb{R}^d \setminus (H \cup Z)} \nu = f \mu \) with \( |f| \leq \lambda \mu \)-a.e., by the Lebesgue-Radon-Nikodym theorem (see [Ma1, pages 36 to 39], for instance).

**Proof of Lemma 5.1.2(b).** Assume first that the family of cubes \( \{Q_j\}_j \) is finite. Then we may suppose that this family of cubes is ordered in such a way that the sizes of the cubes \( R_j \) are non decreasing (i.e. \( \ell(R_{j+1}) \geq \ell(R_j) \)). The functions \( b_j \) that we will construct will be of the form \( b_j = c_j \chi_{A_j} \), with \( c_j \in \mathbb{R} \) and \( A_j \subset R_j \). We set \( A_1 = R_1 \) and \( b_1 := c_1 \chi_{R_1} \), where the constant \( c_1 \) is chosen so that \( \int_{Q_1} w_1 \, d\nu = \int b_1 \, d\mu \).

Suppose that \( b_1, \ldots, b_{k-1} \) have been constructed, satisfy (5.1.5) and \( \sum_{j=1}^{k-1} |b_j| \leq C_0 \lambda \), where \( C_0 \) is some constant which will be fixed below. Let \( R_{s_1}, \ldots, R_{s_m} \) be the subfamily of \( R_1, \ldots, R_{k-1} \) such that \( R_{s_i} \cap R_k \neq \emptyset \). As \( \ell(R_{s_i}) \leq \ell(R_k) \) (because of the non decreasing sizes of \( R_j \)), we have \( R_{s_i} \subset 3R_k \). Taking into account that \( \int |b_j| \, d\mu \leq |\nu|(Q_j) \) for \( j = 1, \ldots, k-1 \) by (5.1.5), and using (5.1.3) and that \( \mu(6R_k) \leq C \mu(R_k) \) (because \( \frac{1}{2} R_k = 3Q_k \) intersects \( \text{supp} \mu \) by (5.1.3)), we get

\[
\sum_i \int |b_{s_i}| \, d\mu \leq \sum_i |\nu|(Q_{s_i}) \leq C|\nu|(3R_k) \leq C \lambda \mu(6R_k) \leq C_2 \lambda \mu(R_k).
\]
Therefore, \( \mu \{ x \in R_k : \sum |b_n(x)| > 2C_2\lambda \} \leq \mu(R_k)/2 \). So, if we set
\[
A_k := \{ x \in R_k : \sum |b_n(x)| \leq 2C_2\lambda \},
\]
then \( \mu(A_k) \geq \mu(R_k)/2 \).

The constant \( c_k \) is chosen so that for \( b_k = c_k \chi_{\Lambda_k} \) we have \( \int b_k \, d\mu = \int_{Q_k} w_k \, d\nu \). Then we obtain, by (5.1.3),
\[
|c_k| \leq \frac{\|\nu\|_{L^1(\mu)}}{\mu(A_k)} \leq \frac{2\|\nu\| R_k}{\mu(R_k)} \leq C_3 \lambda
\]
(this calculation also applies to \( k = 1 \)). Thus, \( |b_k| + \sum |b_n| \leq (2C_2 + C_3) \lambda \). If we choose \( C_0 = 2C_2 + C_3 \), (5.1.7) follows.

Now it is easy to check that (5.1.6) also holds. Indeed we have
\[
\|b_j\|_{L^\infty(\mu)} \mu(R_j) \leq C|c_j| \mu(A_j) = C \left| \int_{Q_j} w_j \, d\nu \right| \leq C\|\nu\|_{L^1(\mu)}.
\]

Suppose now that the collection of cubes \( \{Q_j\}_j \) is not finite. For each fixed \( N \) we consider the family of cubes \( \{Q_j\}_{1 \leq j \leq N} \). Then, as above, we construct functions \( b^N_1, \ldots, b^N_N \) with \( \text{supp}(b^N_j) \subset R_j \) satisfying \( \int b^N_j \, d\mu = \int_{Q_j} w_j \, d\nu \), \( \sum_{j=1}^{N} |b^N_j| \leq C_0 \lambda \) and \( \|b^N_j\|_{L^\infty(\mu)} \mu(R_j) \leq C\|\nu\|_{L^1(\mu)} \). Notice that the sign of \( b^N_j \) equals the sign of \( \int w_j \, d\nu \) and so it does not depend on \( N \).

Then there is a subsequence \( \{b^N_k\}_{k \in I_1} \) which is convergent in the weak * topology of \( L^\infty(\mu) \) to some function \( b_1 \in L^\infty(\mu) \). Now we can consider a subsequence \( \{b^N_k\}_{k \in I_2} \) with \( I_2 \subset I_1 \) which is also convergent in the weak * topology of \( L^\infty(\mu) \) to some function \( b_2 \in L^\infty(\mu) \). In general, for each \( j \) we consider a subsequence \( \{b^N_k\}_{k \in I_j} \) with \( I_j \subset I_{j-1} \) that converges in the weak * topology of \( L^\infty(\mu) \) to some function \( b_j \in L^\infty(\mu) \). It is easily checked that the functions \( b_j \) satisfy the required properties. \qed

### 5.1.2 Proof of Theorem 5.0.12(a)

The proof of Theorem 5.0.12(a) uses the Calderón-Zygmund decomposition developed above and standard arguments. Set \( \mu := \mathcal{H}^n_{\Gamma \cap B} \), where \( \mathcal{H}^n_{\Gamma \cap B} \) is some fixed ball \( B \subset \mathbb{R}^d \). Let \( \nu \in M(\mathbb{R}^d) \) be a finite Radon measure with compact support and \( \lambda > 2^{d+1}\|\nu\|/\|\mu\| \). We will show that
\[
\mu(\{ x \in \mathbb{R}^d : (\nu_{\rho} \circ T_{x}) \nu(x) > \lambda \}) \leq \frac{C}{\lambda} \|\nu\|,
\]
where \( C > 0 \) depends on \( n, d, K, \rho \) and \( \Gamma \), but not on \( B \subset \mathbb{R}^d \). Let us check that (5.1.8) implies that \( \nu_{\rho} \circ T_{x} \) is bounded from \( M(\mathbb{R}^d) \) into \( L_1(\mathcal{H}^n_{\Gamma \cap B}) \). Suppose that \( \nu \) is not compactly supported. Let \( \nu_N = \chi_{B(0,N)} \nu \). Let \( N_0 \) be such that \( \text{supp}\mu \subset B(0, N_0) \). Then it is not hard to show that, for \( x \in \text{supp}\mu \),
\[
|(\nu_{\rho} \circ T_{x}) \nu(x) - (\nu_{\rho} \circ T_{x}) \nu_N(x)| \leq C \frac{|\nu|_{(\mathbb{R}^d \setminus B(0,N))}}{N - N_0},
\]
thus \((\mathcal{V}_\varphi \circ T_\varphi)\nu_N(x) \to (\mathcal{V}_\varphi \circ T_\varphi)\nu(x)\) for all \(x \in \text{supp}\mu\), and since the estimate (5.1.8) holds by assumption for \(\nu_N\), letting \(N \to \infty\), we deduce that it also holds for \(\nu\). Now, by increasing the size of the ball \(B\) and monotone convergence, (5.1.8) yields \(\mathcal{H}_1^\mu\left(\{x \in \mathbb{R}^d : (\mathcal{V}_\varphi \circ T_\varphi)\nu(x) > \lambda\}\right) \leq C\lambda^{-1}\|\nu\|\), as desired. Thus, we only have to verify (5.1.8) for all compactly supported \(\nu\).

Let \(\{Q_j\}_j\) be the almost disjoint family of cubes of Lemma 5.1.2, and set \(\Omega := \bigcup_j Q_j\) and \(R_j := 6Q_j\). Then we can write \(\nu = g\mu + \nu_b\), with

\[
g\mu = \chi_{\mathbb{R}^d \setminus \Omega}\nu + \sum_j b_j\mu \quad \text{and} \quad \nu_b = \sum_j \nu_b^j := \sum_j (w_j \nu - b_j\mu),
\]

where the functions \(b_j\) satisfy (5.1.5), (5.1.6), (5.1.7) and \(w_j = \chi_{Q_j}\left(\sum_k \chi_{Q_k}\right)^{-1}\).

By the subadditivity of \(\mathcal{V}_\varphi \circ T_\varphi\), we have

\[
\mu\left(\{x \in \mathbb{R}^d : (\mathcal{V}_\varphi \circ T_\varphi)\nu(x) > \lambda\}\right)
\leq \mu\left(\{x \in \mathbb{R}^d : (\mathcal{V}_\varphi \circ T_\varphi^\mu)g(x) > \lambda/2\}\right) + \mu\left(\{x \in \mathbb{R}^d : (\mathcal{V}_\varphi \circ T_\varphi)\nu_b(x) > \lambda/2\}\right).
\]

(5.1.9)

Since \(\mathcal{V}_\varphi \circ T_\varphi^\mu\) is bounded in \(L^2(\mathcal{H}_1^\mu)\) by Theorem 5.0.11, it is easy to show that \(\mathcal{V}_\varphi \circ T_\varphi^\mu\) is bounded in \(L^2(\mu)\), with a bound independent of \(B\). Notice that \(|g| \leq C\lambda\) by (5.1.4) and (5.1.7). Then, using (5.1.6),

\[
\mu\left(\{x \in \mathbb{R}^d : (\mathcal{V}_\varphi \circ T_\varphi^\mu)g(x) > \lambda/2\}\right) \leq \frac{1}{\lambda} \int |g| (\mathcal{V}_\varphi \circ T_\varphi^\mu)g \, d\mu \leq \frac{1}{\lambda} \int |g|^2 \, d\mu
\]

\[
\leq \frac{1}{\lambda} \int |g| \, d\mu \leq \frac{1}{\lambda}\left(|\nu|_{(\mathbb{R}^d \setminus \Omega)} + \sum_j \int_{R_j} |b_j| \, d\mu\right)
\]

\[
\leq \frac{1}{\lambda}\left(|\nu|_{(\mathbb{R}^d \setminus \Omega)} + \sum_j |\nu|(Q_j)\right) \leq \frac{C}{\lambda} \|\nu\|.
\]

(5.1.10)

Let \(\hat{\Omega} := \bigcup_j 2Q_j\). By (5.1.2), we have \(\mu(\hat{\Omega}) \leq \sum_j \mu(2Q_j) \lesssim \lambda^{-1} \sum_j |\nu|(Q_j) \lesssim \lambda^{-1}\|\nu\|\). We are going to show that

\[
\mu\left(\{x \in \mathbb{R}^d \setminus \hat{\Omega} : (\mathcal{V}_\varphi \circ T_\varphi)\nu_b(x) > \lambda/2\}\right) \leq \frac{C}{\lambda} \|\nu\|
\]

and then (5.1.8) is a direct consequence of (5.1.9), (5.1.10), (5.1.11) and the estimate \(\mu(\hat{\Omega}) \lesssim \lambda^{-1}\|\nu\|\). Since \(\mathcal{V}_\varphi \circ T_\varphi\) is sublinear,

\[
\mu\left(\{x \in \mathbb{R}^d \setminus \hat{\Omega} : (\mathcal{V}_\varphi \circ T_\varphi)\nu_b(x) > \lambda/2\}\right) \leq \frac{1}{\lambda} \sum_j \int_{\mathbb{R}^d \setminus 2R_j} (\mathcal{V}_\varphi \circ T_\varphi)\nu_b^j \, d\mu
\]

\[
\leq \frac{1}{\lambda} \sum_j \int_{\mathbb{R}^d \setminus 2R_j} (\mathcal{V}_\varphi \circ T_\varphi)\nu_b^j \, d\mu + \frac{1}{\lambda} \sum_j \int_{2R_j \cup 2Q_j} (\mathcal{V}_\varphi \circ T_\varphi)\nu_b^j \, d\mu.
\]

(5.1.12)
We are going to estimate the two terms on the right of (5.1.12) separately. Let us start with the first one. Given \(j\) and \(x \in \text{supp} \mu \setminus 2R_j\), let \(\{\epsilon_m\}_{m \in \mathbb{Z}}\) be a decreasing sequence of positive numbers (which depends on \(j\) and \(x\), i.e. \(\epsilon_m \equiv \epsilon_m(j,x)\)) such that

\[
(V_\rho \circ T_\varphi)\nu_b^j(x) \leq 2 \left( \sum_{m \in \mathbb{Z}} |(K\varphi_{\epsilon_m}^m \ast \nu_b^j)(x)|^\rho \right)^{1/\rho}.
\]  

(5.1.13)

As before, if we set \(I_k := [2^{k-1}, 2^{-k})\), we can decompose \(Z = S \cup L\), where

\[
L := \{m \in \mathbb{Z} : \epsilon_m \in I_k, \epsilon_m+1 \in I_i, \text{ for } i > k\},
\]

\[
S := \bigcup_{k \in \mathbb{Z}} S_k, \quad S_k := \{m \in \mathbb{Z} : \epsilon_m, \epsilon_{m+1} \in I_k\}.
\]

Let \(z_j\) denote the center of \(Q_j\) (and of \(R_j\)). Then, since \(\nu_b^j(R_j) = 0\) and \(\text{supp} \nu_b^j \subset R_j\),

\[
|(K\varphi_{\epsilon_m+1}^m \ast \nu_b^j)(x)| = \left| \int \varphi_{\epsilon_m+1}^m(x-y)K(x-y) \, d\nu_b^j(y) \right|
\]

\[
\leq \int |\varphi_{\epsilon_m+1}^m(x-y)K(x-y) - \varphi_{\epsilon_m}^m(z_j)K(x-z_j)| \, d|\nu_b^j|(y).
\]

(5.1.14)

If \(m \in L\), it is easy to see that \(|\nabla (\varphi_{\epsilon_m+1}^m K)(t)| \lesssim |t|^{-n-1}\). Moreover, since \(x \in \mathbb{R}^d \setminus 2R_j\) and \(\text{supp} \nu_b^j \subset R_j\), there are finitely many \(m \in L\) (which depends only on \(n\) and \(d\)) such that \(\text{supp} \varphi_{\epsilon_m+1}^m (x-\cdot) \cap R_j \neq \emptyset\); hence, there are finitely many \(m \in L\) such that \((K\varphi_{\epsilon_m+1}^m \ast \nu_b^j)(x) \neq 0\), and this number only depends in \(n\) and \(d\). On the other hand, if \(m \in S_k\), it is easy to check that \(|\nabla (\varphi_{\epsilon_m+1}^m K)(t)| \lesssim 2^k |\epsilon_m - \epsilon_{m+1}| |t|^{-n-1}\) (see (3.4.6)). Similarly to the case \(m \in L\), there are finitely many \(k \in \mathbb{Z}\) such that \(\text{supp} \varphi_{2^{-k-1}}^{2-k} (x-\cdot) \cap R_j \neq \emptyset\), and the number only depends on \(n\) and \(d\) (notice that \(\text{supp} \varphi_{\epsilon_m+1}^m (x-\cdot) \subset \text{supp} \varphi_{2^{-k-1}}^{2-k} (x-\cdot)\) for all \(m \in S_k\)).

Using these estimates and remarks, (5.1.13), (5.1.14), and that \(\rho > 2\), we obtain

\[
(V_\rho \circ T_\varphi)\nu_b^j(x) \lesssim \sum_{k \in \mathbb{Z}} \sum_{m \in S_k} |(K\varphi_{\epsilon_m+1}^m \ast \nu_b^j)(x)| + \sum_{m \in L} |(K\varphi_{\epsilon_m+1}^m \ast \nu_b^j)(x)|
\]

\[
\lesssim \sum_{k \in \mathbb{Z} : \text{supp} \varphi_{2^{-k-1}}^{2-k} (x-\cdot) \cap R_j \neq \emptyset} \sum_{m \in S_k} 2^k |\epsilon_m - \epsilon_{m+1}| |x-z_j|^{-n-1} \ell(R_j) \|\nu_b^j\|
\]

\[
+ \sum_{m \in L : \text{supp} \varphi_{\epsilon_m+1}^m (x-\cdot) \cap R_j \neq \emptyset} |x-z_j|^{-n-1} \ell(R_j) \|\nu_b^j\| \lesssim |x-z_j|^{-n-1} \ell(R_j) \|\nu_b^j\|
\]

for all \(j\) and \(x \in \text{supp} \mu \setminus 2R_j\). Therefore, using that \(\mu\) is AD regular on the \(R_j\)'s, that \(\|\nu_b^j\| \lesssim |\nu|(Q_j)\), that the \(Q_j\)'s are semidisjoint, and standard arguments,

\[
\sum_j \int_{\mathbb{R}^d \setminus 2R_j} (V_\rho \circ T_\varphi)\nu_b^j \, d\mu \lesssim \sum_j \ell(R_j) \|\nu_b^j\| \int_{\mathbb{R}^d \setminus 2R_j} |x-z_j|^{-n-1} \, d\mu \lesssim \sum_j \|\nu_b^j\| \lesssim \|\nu\|.
\]

(5.1.15)
5.2. $L^p$ boundedness of $V_\rho \circ T^\mu_\varphi$ for uniformly rectifiable measures $\mu$

Let us now estimate the second term on the right hand side of (5.1.12). As before, given $j$ and $x \in 2R_j \setminus 2Q_j$, let $\{\varepsilon_m\}_{m \in \mathbb{Z}}$ be a decreasing sequence of positive numbers (which depends on $j$ and $x$, i.e. $\varepsilon_m = \varepsilon_m(j, x)$) such that

$$ (V_\rho \circ T_\varphi)(w_j \nu)(x) \leq 2 \left( \sum_{m \in \mathbb{Z}} |(K_{\varphi_{\varepsilon_{m+1}}} * (w_j \nu))(x)|^\rho \right)^{1/\rho}. $$

Since $\rho > 2$, $V_\rho \circ T_\varphi$ is sublinear, and $\nu_j^i = w_j \nu - b_j \mu$, for $x \in 2R_j \setminus 2Q_j$ we have

$$ (V_\rho \circ T_\varphi)\nu_j^i(x) \leq (V_\rho \circ T_\varphi)(w_j \nu)(x) + (V_\rho \circ T)(b_j \mu)(x) $$

$$ \leq 2 \sum_{m \in \mathbb{Z}} \left| (K_{\varphi_{\varepsilon_{m+1}}} * (w_j \nu))(x) \right| + (V_\rho \circ T^\mu_\varphi)b_j(x) $$

$$ \lesssim |\nu|(Q_j) |x - z_j|^{-n} + (V_\rho \circ T^\mu_\varphi)b_j(x), $$

where $z_j$ denotes the center of $Q_j$, as before. Since $V_\rho \circ T^\mu_\varphi$ is a bounded operator in $L^2(\mu)$ by Theorem 5.0.11, using the last estimate and Cauchy-Schwarz we have

$$ \sum_j \int_{2R_j \setminus 2Q_j} (V_\rho \circ T_\varphi)\nu_j^i d\mu \lesssim \sum_j \int_{2R_j \setminus 2Q_j} \frac{|\nu|(Q_j)}{|x - z_j|^n} d\mu(x) + \sum_j \int_{2R_j \setminus 2Q_j} (V_\rho \circ T^\mu_\varphi)b_j d\mu $$

$$ \lesssim \sum_j |\nu|(Q_j) \frac{\mu(2R_j)}{|(Q_j)|^n} + \sum_j \|V_\rho \circ T^\mu_\varphi\|_{L^2(\mu)}b_j \frac{\mu(2R_j)}{1/2} $$

$$ \lesssim \sum_j |\nu|(Q_j) + \sum_j \|b_j\|_{L^\infty(\mu)} \mu(R_j) \lesssim \sum_j |\nu|(Q_j) \lesssim \|\nu\|. $$

Finally, applying this estimate and (5.1.15) to (5.1.12), we obtain (5.1.11), and Theorem 5.0.12(a) is proved.

5.2 $L^p$ boundedness of $V_\rho \circ T^\mu_\varphi$ for uniformly rectifiable measures $\mu$

The purpose of this section is to prove the following theorem and the corollary that it yields.

**Theorem 5.2.1.** Let $E \subset \mathbb{R}^d$ be an $n$-dimensional AD regular set. Let $\rho > 2$. Assume that there exist constants $C_0$ and $C_1$ such that, for each ball $B$ centered on $E$, there is a set $F = F_B$ such that:

(a) $\mathcal{H}^n_F(F \cap B) \geq C_0 \mathcal{H}^n(B)$,

(b) $V_\rho \circ T_\varphi$ is bounded from $M(\mathbb{R}^d)$ to $L^1(\mathcal{H}^n_F)$ with constant bounded by $C_1$.

Then $V_\rho \circ T_\varphi$ is bounded from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mathcal{H}^n_F)$, and $V_\rho \circ T^\mu_\varphi$ is a bounded operator in $L^p(\mathcal{H}^n_F)$ for all $1 < p < \infty$. 

The following corollary of the previous theorem and Theorem 5.0.12 is the main result of this section.

**Corollary 5.2.2.** If $E$ is an $n$-dimensional AD regular uniformly rectifiable set, then $\mathcal{V}_{\rho} \circ \mathcal{T}^{H_E}_\varphi$ is a bounded operator in $L^p(H^n_E)$ for all $1 < p < \infty$ and $\rho > 2$. Moreover, the operator $\mathcal{V}_{\rho} \circ \mathcal{T}^{H_E}_\varphi$ is bounded from $M(\mathbb{R}^d)$ to $L^{1,\infty}(H^n_E)$.

**Proof.** Recall from [DS2, Definition 1.26] that a set $E$ has BPLG (big pieces of Lipschitz graphs) if $E$ is $n$-AD regular and if there exist constants $C_1 > 0$ and $\theta > 0$ such that, for any $x \in E$ and $r > 0$, there is (a rotation and translation of) an $n$-dimensional Lipschitz graph $\Gamma$ with constant less than $C_1$ such that $H^n_E(\Gamma \cap B(x,r)) \geq \theta r^n$. Thus, if $E$ has BPLG, the assumption (a) of Theorem 5.2.1 are satisfied by taking $F = \Gamma$, and Theorem 5.0.12 gives us the assumption (b) with uniform constant (since the slope of the Lipschitz graphs is uniformly bounded), so Theorem 5.2.1 can be applied. Therefore, we deduce that, if $E$ has BPLG and $\rho > 2$, then $\mathcal{V}_{\rho} \circ \mathcal{T}^{H_E}_\varphi$ is bounded from $M(\mathbb{R}^d)$ to $L^{1,\infty}(H^n_E)$, and $\mathcal{V}_{\rho} \circ \mathcal{T}^{H^n_E}_\varphi$ is a bounded operator in $L^p(H^n_E)$ for all $1 < p < \infty$.

Similarly, a set $E$ has (BP)$^2$LG (big pieces of big pieces of Lipschitz graphs) if there exist constants $C_g$, $\theta$, and $0 < \alpha \leq 1$ so that, if $B$ is any ball centered on $E$, then there is an $n$-dimensional AD regular set $F \subset \mathbb{R}^d$ (with constant bounded by $C_g$) such that $\mu(F \cap B) \geq \alpha \mu(B)$ and, for each $x \in F$ and $r > 0$, there is an $n$-dimensional Lipschitz graph, say $\Gamma$, with slope bounded by $C_g$ and such that $H^n(F \cap \Gamma \cap B(x,r)) \geq \theta r^n$. Notice that, any of those sets $F$ have BPLG, so the variation operator is bounded on the corresponding spaces, by the previous comments. Hence, we can apply once again Theorem 5.2.1 to $E$ (now (b) is satisfied for the big pieces $F$ of $E$), and we deduce that, for any set $E$ which has (BP)$^2$LG, $\mathcal{V}_{\rho} \circ \mathcal{T}^{H_E}_\varphi$ is bounded from $M(\mathbb{R}^d)$ to $L^{1,\infty}(H^n_E)$, and $\mathcal{V}_{\rho} \circ \mathcal{T}^{H^n_E}_\varphi$ is a bounded operator in $L^p(H^n_E)$ for all $1 < p < \infty$.

Finally, from [DS2, page 22] and the remark given in [DS2, page 16], we know that if $E$ is a uniformly rectifiable set then $E$ has (BP)$^2$LG. Therefore, the corollary is proved.

### 5.2.1 Dyadic cubes in the sense of [DS2, Chapter 3 of Part I]

For the study of the uniformly rectifiable sets we will use the “dyadic cubes” built by G. David in [Da1, Appendix 1] (see also [DS2, Chapter 3 of Part I]). These dyadic cubes are not true cubes, but they play this role with respect the $n$-dimensional AD regular set $E$, in a sense. Let us explain which are the precise results and properties about this “dyadic lattice”. For each $j \in \mathbb{Z}$, there exists a family $\mathcal{D}_j$ of Borel subsets of $E$ such that:

(a) each $\mathcal{D}_j$ is a partition of $E$, i.e. $E = \bigcup_{Q \in \mathcal{D}_j} Q$ and $Q \cap Q' = \emptyset$ whenever $Q, Q' \in \mathcal{D}_j$ and $Q \neq Q'$.
Lemma 5.2.3. If $Q \in \mathcal{D}_j$ and $Q' \in \mathcal{D}_k$ with $k \leq j$, then either $Q \subset Q'$ or $Q \cap Q' = \emptyset$;

c for all $j \in \mathbb{Z}$ and $Q \in \mathcal{D}_j$, we have $2^{-j} \lesssim \text{diam}(Q) \leq 2^{-j}$ and $\mathcal{H}^n_Q(Q) \approx 2^{-jn}$;

d there exists $C > 0$ such that, for all $j \in \mathbb{Z}$, $Q \in \mathcal{D}_j$, and $0 < \tau < 1$, 
$\mathcal{H}^n_Q \left( \{ x \in Q : \text{dist}(x, E \setminus Q) \leq \tau 2^{-j} \} \right) + \mathcal{H}^n_Q \left( \{ x \in E \setminus Q : \text{dist}(x, Q) \leq \tau 2^{-j} \} \right) \leq Ct^{1/C}2^{-jn}$. (5.2.1)

This is commonly known as the small boundaries condition. As a consequence (see [DS2, Lemma 3.5 of Part I]), there is a point $z_Q \in Q$ (the center of $Q$) such that $\text{dist}(z_Q, E \setminus Q) \gtrsim 2^{-j}$.

We denote $\mathcal{D} := \bigcup_{j \in \mathbb{Z}} \mathcal{D}_j$. For $Q \in \mathcal{D}_j$, we define the side length of $Q$ as $\ell(Q) = 2^{-j}$. Notice that $\ell(Q) \lesssim \text{diam}(Q) \leq \ell(Q)$. Actually it may happen that a cube $Q$ belongs to $\mathcal{D}_j \cap \mathcal{D}_k$ with $j \neq k$. In this case, $\ell(Q)$ is not well defined. However, this problem can be solved in many ways. For example, the reader may think that a cube is not only a subset of $E$, but a couple $(Q, j)$, where $Q$ is a subset of $E$ and $j \in \mathbb{Z}$ is such that $Q \in \mathcal{D}_j$.

From here till the end of the chapter, $\mathcal{D}$ will denote a fixed dyadic lattice with respect to a given $n$-dimensional AD regular set in the sense of [DS2]. Given an $n$-dimensional AD regular measure $\mu$ (notice that, by Lebesgue’s differentiation theorem, $\mu = h \mathcal{H}^n_{\text{supp}\mu}$ for some $h \approx 1$), $\mathcal{D}$ will denote the dyadic lattice with respect to $\text{supp}\mu$. Given $\lambda > 1$ and $Q \in \mathcal{D}$, we will also set $\lambda Q := \{ x \in E : \text{dist}(x, Q) \leq (\lambda - 1)\ell(Q) \}$.

Observe that $\text{diam}(\lambda Q) \leq \text{diam}(Q) + 2(\lambda - 1)\ell(Q) \leq (2\lambda - 1)\ell(Q)$.

### 5.2.2 Proof of Theorem 5.2.1

Our proof of Theorem 5.2.1 uses a Whitney’s decomposition of $E \cap \Omega$, where $\Omega \subset \mathbb{R}^d$ is a given open set, in terms of the cubes $Q \in \mathcal{D}$. In the next lemma we state the precise version of the decomposition we need. The proof of the lemma, which we will omit, is almost the same as in the classical Whitney decomposition of an open set of $\mathbb{R}^d$, but replacing the standard dyadic lattice of $\mathbb{R}^d$ by the special dyadic lattice $\mathcal{D}$ associated to $\mathcal{H}^n_Q$.

Lemma 5.2.3. If $\Omega \subset \mathbb{R}^d$ is open, $\Omega \neq \mathbb{R}^d$, then $E \cap \Omega$ can be decomposed as $E \cap \Omega = \bigcup_{i \in I} Q_i$, where $I \subset \mathbb{N}$, $Q_i \in \mathcal{D}$ for all $i \in I$, for some constants $R > 20$ and $D_0 \geq 1$, the following holds,

(a) $Q_k \cap Q_j = \emptyset$ if $k \neq j$.

(b) $10Q_k \subset \Omega$ and $RQ_k \cap \Omega^c \neq \emptyset$.

(c) For each $Q_k$ with $k \in I$, there are at most $D_0$ sets $Q_j$ with $j \in I$ such that $10Q_k \cap 10Q_j \neq \emptyset$. Moreover, for such $Q_k$, $Q_j$, we have $\ell(Q_k) \approx \ell(Q_j)$. 


Proof of Theorem 5.2.1. Fix $\rho > 2$ and set $\mu = \mathcal{H}^d_\rho$ for some $n$-dimensional AD regular uniformly rectifiable set $E \subset \mathbb{R}^d$ satisfying the assumptions of the theorem. We are going to use a good $\lambda$ inequality: we will show that there exists some absolute constant $\eta > 0$ such that for all $\epsilon > 0$ there exists $\delta := \delta(\epsilon) > 0$ such that
\[
\mu \left( \left\{ x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu(x) > (1 + \epsilon)\lambda, \ M^\mu \nu(x) \leq \delta \lambda \right\} \right) \\
\leq (1 - \eta)\mu \left( \left\{ x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu(x) > \lambda \right\} \right) \tag{5.2.2}
\]
for all $\lambda > 0$ and $\nu \in M(\mathbb{R}^d)$, where $M^\mu \nu(x) := \text{supp}_{r>0}\nu(B(x,r))/\mu(B(x,r))$ for $x \in \text{supp}\mu$.

It is easy to check that this implies that $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$ is bounded from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mu)$, and that $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu$ is a bounded operator in $L^p(\mu)$ for all $1 < p < \infty$, by standard arguments (recall that $M^\mu$ is bounded in these spaces).

To prove (5.2.2), consider a Whitney decomposition of $\text{supp}\mu \cap \Omega_\lambda = \bigcup_{i \in I} Q_i$ as in Lemma 5.2.3, where
\[
\Omega_\lambda = \left\{ x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu(x) > \lambda \right\}.
\]
Obviously, we can assume that $\mu(\Omega_\lambda) < \infty$, otherwise (5.2.2) follows. Consider a cube $Q_i$, $i \in I$. The following claim will be proved later on.

Claim 5.2.4. If $x \in Q_i$ satisfies $(\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu(x) > (1 + \epsilon)\lambda$ and $M^\mu \nu(x) \leq \delta \lambda$, then
\[
(\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{2Q_i}\nu)(x) > \epsilon \lambda/2, \tag{5.2.3}
\]
assuming that $\delta$ is small enough depending on $\epsilon$.

Let $G_i$ be a ball centered on $Q_i$ and such that $G_i \cap \text{supp}\mu \subset Q_i$ and $\mu(G_i) \approx \mu(Q_i)$, and let $F_i$ denote the AD regular set related to $G_i$ given by the assumptions of the theorem. Set $\mu_i := \mathcal{H}^d_{\rho_i}$, so $\mu|_{Q_i \cap F_i} = \mu_i|_{Q_i \cap E}$. Since the operator $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$ is bounded from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mu_i)$ for all $i \in S$ with uniform bounds by hypothesis, using (5.2.3) we deduce
\[
\mu \left( x \in Q_i \cap F_i : (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu(x) > (1 + \epsilon)\lambda, \ M^\mu \nu(x) \leq \delta \lambda \right) \\
\leq \mu \left( x \in Q_i \cap F_i : (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{2Q_i}\nu)(x) > \epsilon \lambda/2 \right) \\
= \mu_i \left( x \in Q_i \cap E : (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{2Q_i}\nu)(x) > \epsilon \lambda/2 \right) \\
\leq \mu_i \left( x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{2Q_i}\nu)(x) > \epsilon \lambda/2 \right) \leq C(\epsilon \lambda)^{-1}\|\chi_{2Q_i}\nu\|.
\]
Notice that, if $Q_i$ contains some point $x$ such that $M^\mu \nu(x) \leq \delta \lambda$, then
\[
\|\chi_{2Q_i}\nu\| \leq \int_{B(x,2\ell(Q_i))} d|\nu| \leq \mu(B(x,2\ell(Q_i))) M^\mu \nu(x) \leq \mu(5Q_i) M^\mu \nu(x) \leq C\delta \lambda \mu(Q_i).
\]
Thus, $\mu \left( x \in Q_i \cap F_i : (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu(x) > (1 + \epsilon)\lambda, \ M^\mu \nu(x) \leq \delta \lambda \right) \leq C_3 \delta \epsilon^{-1} \mu(Q_i)$, where $C_3$ does not depend on $i \in I$. Using this last estimate for all the $Q_i$’s with $i \in I$, and since
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$\mu(Q_i) \geq \mu(Q_i \cap F_i) = \mu(G_i \cap F_i) \geq C_0 \mu(G_i) \geq CC_0 \mu(Q_i)$ (notice that $C_4 := CC_0 \leq 1$).

$$\mu\{x \in \mathbb{R}^d : (\nu \circ T_\varphi)\nu(x) > (1 + \varepsilon)\lambda, \ M^\mu\nu(x) \leq \delta\lambda\} \leq \sum_{i \in I} \mu(Q_i \setminus F_i)$$

$$+ \sum_{i \in I} \mu\{x \in Q_i \cap F_i : (\nu \circ T_\varphi)\nu(x) > (1 + \varepsilon)\lambda, \ M^\mu\nu(x) \leq \delta\lambda\}$$

$$\leq \sum_{i \in I} (1 - C_4) \mu(Q_i) + \sum_{i \in I} C_3 \delta^{-1} \mu(Q_i) \leq (1 - (C_4 - C_3 \delta^{-1})) \mu(\Omega_\lambda).$$

If we choose $\delta = \delta(\varepsilon)$ such that $\delta \leq C_4 (2C_3)^{-1} \varepsilon$, then $C_4 - C_3 \delta^{-1} \delta \geq C_4/2$, and we finally obtain $\mu\{x \in \mathbb{R}^d : (\nu \circ T_\varphi)\nu(x) > (1 + \varepsilon)\lambda, \ M^\mu\nu(x) \leq \delta\lambda\} \leq (1 - C_4/2) \mu(\Omega_\lambda)$. Hence, (5.2.2) follows by taking $0 < \eta = C_4/2 < 1$.  

**Proof of Claim 5.2.4.** Take $z \in RQ_i \setminus \Omega_\lambda$, so that $(\nu \circ T_\varphi)\nu(z) \leq \lambda$. Since $x, z \in RQ_i \subset B(z, 2R\ell(Q_i)) =: B_i$ and $\nu \circ T_\varphi$ is sublinear and positive, by standard computations,

$$\|(\nu \circ T_\varphi)(\chi_{R^d \setminus 2B_i} \nu)(x) - (\nu \circ T_\varphi)(\chi_{R^d \setminus 2B_i} \nu)(z)\|$$

$$\leq \sup_{\varepsilon_m} \left( \sum_{m \in \mathbb{Z}} |(K \varphi_{\varepsilon_{m+1}}^m * (\chi_{R^d \setminus 2B_i} \nu))(x) - (K \varphi_{\varepsilon_{m+1}}^m * (\chi_{R^d \setminus 2B_i} \nu))(z)|^\rho \right)^{1/\rho}$$

$$\leq \sup_{\varepsilon_m} \left( \sum_{m \in \mathbb{Z}} \left( \int_{B_i^{m}(x,z)} |\nabla (\varphi_{\varepsilon_{m+1}}^m K)(u_{x,z}(y) - y)| \, |x - y| \, d\nu(y) \right)^\rho \right)^{1/\rho},$$

where $B_i^{m}(x, z) := (\mathbb{R}^d \setminus 2B_i) \cap (\text{supp} \varphi_{\varepsilon_{m+1}}^m (x - \cdot) \cup \text{supp} \varphi_{\varepsilon_{m+1}}^m (z - \cdot))$ and $u_{x,z}(y)$ is some point lying on the segment joining $x$ and $z$. For each $x$ and $z$, let $\varepsilon_m \equiv \varepsilon_m(x, z)$ be a sequence that takes the supremum in the right hand side of the last inequality in (5.2.4). Given $\varepsilon_m > 0$, let $j(\varepsilon_m)$ denote the integer such that $\varepsilon_m \in [2^{-j(\varepsilon_m)-1}, 2^{-j(\varepsilon_m)})$. For $j \in \mathbb{Z}$ we set $I_j := [2^{-j-1}, 2^{-j})$. We decompose $\mathbb{Z} = S \cup \mathcal{L}$, where $S := \bigcup_{j \in \mathbb{Z}} S_j$, $S_j := \{m \in \mathbb{Z} : \varepsilon_m, \varepsilon_{m+1} \in I_j\}$, $\mathcal{L} := \{m \in \mathbb{Z} : \varepsilon_m \in I_i, \varepsilon_{m+1} \in I_j \text{ for } i < j\}$.

Notice that if $2^{-j+2} < r(B_i)$, where $r(B_i)$ denotes the radius of $B_i$, then $B_i^{m}(x, z) = \emptyset$ for all $m \in S_j$. Therefore, we can assume that $j \leq \log_2(4/r(B_i))$. If $m \in S_j$, then $B_i^{m}(x, z) \subset B(x, 2^{-j+3})$, and for $t \in \text{supp} \varphi_{\varepsilon_{m+1}}^m K$ we have that $|\nabla (\varphi_{\varepsilon_{m+1}}^m K)(t)| \lesssim 2^{(n+2)|\varepsilon_m - \varepsilon_{m+1}|}$ (see (3.4.6)). If $m \in \mathcal{L}$, we easily have $|\nabla (\varphi_{\varepsilon_{m+1}}^m K)(t)| \lesssim |t|^{-n-1}$. Therefore, using (5.2.4), that $\rho > 2$, that the sets $B_i^{m}(x, z)$ have bounded overlap for $m \in \mathcal{L}$, and that $|x - z| \lesssim r(B_i)$, we
have

\[ |(\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{\mathbb{R}^d \setminus 2B_i, \nu})(x) - (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{\mathbb{R}^d \setminus 2B_i, \nu})(z)| \]

\[ \lesssim \sum_{j \leq \log_2(4/r(B_i))} \sum_{m \in S_j} |x - z|^{2j(n+2)} |\epsilon_m - \epsilon_{m+1}| \int_{B(x, 2^{-j+3})} \frac{d|\nu|(y)}{|x - y|^{n+1}} \]

\[ + |x - z| \sum_{m \in L} \int_{B_m(x, z)} |x - y|^{-n-1} \frac{d|\nu|(y)}{|x - y|^{n+1}} \]

\[ \lesssim \sum_{j \leq \log_2(4/r(B_i))} r(B_i) 2^{j(n+2)} \int_{B(x, 2^{-j+3})} d|\nu|(y) + r(B_i) \int_{\mathbb{R}^d \setminus 2B_i} \frac{d|\nu|(y)}{|x - y|^{n+1}} \]

\[ \lesssim \sum_{j \leq \log_2(4/r(B_i))} \frac{r(B_i) 2^j}{\mu(B(x, 2^{-j+3}))} \int_{B(x, 2^{-j+3})} d|\nu|(y) \]

\[ + r(B_i) \sum_{k \geq 1} \int_{2^{k+2}r(B_i) \geq |x - y| \geq 2^{k-1}r(B_i)} \frac{d|\nu|(y)}{|x - y|^{n+1}} \]

\[ \lesssim M^\nu(x) + \sum_{k \geq 1} \frac{2^{-k}}{\mu(B(x, 2^{k+2}r(B_i)))} \int_{B(x, 2^{k+2}r(B_i))} d|\nu|(y) \leq CM^\nu(x). \]

Recall that $2B_i$ is a ball centered at $z \notin \Omega_\lambda$. Let \( \{\delta_m\}_{m \in \mathbb{Z}} \) be a decreasing sequence of positive numbers such that

\[ (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{\mathbb{R}^d \setminus 2B_i, \nu})(z) \leq (1 + \epsilon/4) \left( \sum_{m \in \mathbb{Z}} |(K \varphi_{\delta_m} \ast (\chi_{\mathbb{R}^d \setminus 2B_i, \nu}))(z)|^\rho \right)^{1/\rho} \]

\[ \text{(5.2.6)} \]

and let $m_0$ be the integer such that $\delta_{m_0 + 1} < 8r(B_i) \leq \delta_{m_0}$. We can add the term $8r(B_i)$ to the sequence $\{\delta_m\}_{m \in \mathbb{Z}}$ in between $\delta_{m_0}$ and $\delta_{m_0 + 1}$ (if $\delta_{m_0} = 8r(B_i)$ this step is not necessary), and then, by (5.2.6) and the triangle inequality,

\[ (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{\mathbb{R}^d \setminus 2B_i, \nu})(z) \leq (1 + \epsilon/4) \left( \sum_{m \neq m_0} |(K \varphi_{\delta_m} \ast (\chi_{\mathbb{R}^d \setminus 2B_i, \nu}))(z)|^\rho \right)^{1/\rho} \]

\[ + |(K \varphi_{8r(B_i)} \ast (\chi_{\mathbb{R}^d \setminus 2B_i, \nu}))(z) + (K \varphi_{8r(B_i)} \ast (\chi_{\mathbb{R}^d \setminus 2B_i, \nu}))(z)|^\rho \]

\[ \leq (1 + \epsilon/4) \left( |(K \varphi_{8r(B_i)} \ast (\chi_{\mathbb{R}^d \setminus 2B_i, \nu}))(z)| + \sum_{m > m_0} |(K \varphi_{\delta_m} \ast (\chi_{\mathbb{R}^d \setminus 2B_i, \nu}))(z)| \right) \]

\[ + \left( \sum_{m < m_0} |(K \varphi_{\delta_m} \ast (\chi_{\mathbb{R}^d \setminus 2B_i, \nu}))(z)|^\rho + |(K \varphi_{8r(B_i)} \ast (\chi_{\mathbb{R}^d \setminus 2B_i, \nu}))(z)|^\rho \right)^{1/\rho}. \]

On one hand, $\text{supp}(\varphi_{\delta_m}) (z - \cdot) \subset \mathbb{R}^d \setminus 2B_i$ for all $m < m_0$, and the same holds for $\varphi_{8r(B_i)}$. 

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Thus,

\[
\left( \sum_{m < m_0} |(K \varphi_{\delta_{m+1}}^m * (\chi_{\mathbb{R}^d \setminus 2B_0})(z))|^p + |(K \varphi_{\delta_{m_0}}^m * (\chi_{\mathbb{R}^d \setminus 2B_0})(z))|^p \right)^{1/p} = \left( \sum_{m < m_0} |(K \varphi_{\delta_{m+1}}^m * \nu)(z)|^p + |(K \varphi_{\delta_{m_0}}^m * \nu)(z)|^p \right)^{1/p} \leq (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu(z) \leq \lambda.
\]

(5.2.8)

On the other hand,

\[
|(K \varphi_{\delta_{m+1}}^m * (\chi_{\mathbb{R}^d \setminus 2B_0})(z))| + \sum_{m > m_0} |(K \varphi_{\delta_{m+1}}^m * (\chi_{\mathbb{R}^d \setminus 2B_0})(z))| \lesssim \int_{\mathbb{R}^d \setminus 2B_0} \left( \varphi_{\delta_{m+1}}^m(z-y) + \sum_{m > m_0} \varphi_{\delta_{m+1}}^m(z-y) \right) |z-y|^{-n} d\nu(y) \lesssim \int_{\mathbb{R}^d \setminus 2B_0} \chi_{B(z,32r(B_0))}(y)r(B_0)^{-n} d\nu(y) \lesssim \int_{B(x,33r(B_0))} \mu(B(x,33r(B_0)))^{-1} d\nu(y) \leq M^\mu \nu(x).
\]

(5.2.9)

Applying (5.2.8) and (5.2.9) to (5.2.7), we deduce \((\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{\mathbb{R}^d \setminus 2B_0})(z) \leq (1 + \epsilon/4)(\lambda + CM^\mu \nu(x))\). Also, similarly to (5.2.9), since \(x \in Q_1\), one can check that \((\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{2B_1 \setminus 2Q})(x) \leq CM^\mu \nu(x)\). Therefore, combining this two last estimates with (5.2.5) and that \(M^\mu \nu(x) \leq \delta \lambda\),

\[
(\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{\mathbb{R}^d \setminus 2Q})(x) \leq (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{2B_1 \setminus 2Q})(x) + (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{\mathbb{R}^d \setminus 2B_0})(x) \leq (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{2B_1 \setminus 2Q})(x) + (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{\mathbb{R}^d \setminus 2B_0})(x) + |(\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{\mathbb{R}^d \setminus 2B_0})(x) - (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{\mathbb{R}^d \setminus 2B_0})(z)| \leq CM^\mu \nu(x) + (1 + \epsilon/4)(\lambda + CM^\mu \nu(x)) + CM^\mu \nu(x) \leq \left((1 + \epsilon/4) + (3 + \epsilon/4)C\delta\right)\lambda,
\]

and thus, since \(\mathcal{V}_\rho \circ \mathcal{T}_\varphi\) is sublinear and positive,

\[
(\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{2Q})(x) \geq (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{2Q})(x) - (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{\mathbb{R}^d \setminus 2Q})(x) > (1 + \epsilon)\lambda - \left((1 + \epsilon/4) + (3 + \epsilon/4)C\delta\right)\lambda = (3\epsilon/4 - (3 + \epsilon/4)C\delta)\lambda.
\]

If we choose \(\delta \leq \epsilon C^{-1}(12 + \epsilon)^{-1}\), then \((3\epsilon/4 - (3 + \epsilon/4)C\delta)\lambda \geq \epsilon\lambda/2\) and the claim follows.

5.3 \( L^2 \) boundedness of \( \mathcal{V}_\rho \circ \mathcal{T}^\mu \) for uniformly rectifiable measures \( \mu \)

This section is devoted to prove the following theorem, whose proof is based on the techniques developed in Section 4.1 and a corona decomposition.
Theorem 5.3.1. Let $\rho > 2$ and let $\mu$ be an $n$-dimensional AD regular Borel measure on $\mathbb{R}^d$. If $\mu$ is uniformly rectifiable, then $\mathcal{V}_\rho \circ \mathcal{T}^\mu$ is bounded in $L^2(\mu)$.

Given $j \in \mathbb{Z}$, set $I_j := [2^{-j-1}, 2^{-j})$. Then, using the triangle inequality, we can split the variation operator into the short variation and the long variation, i.e., $(\mathcal{V}_\rho \circ \mathcal{T}^\mu)f(x) \leq (\mathcal{V}_\rho^S \circ \mathcal{T}^\mu)f(x) + (\mathcal{V}_\rho^L \circ \mathcal{T}^\mu)f(x)$, where we have set

\[
(\mathcal{V}_\rho^S \circ \mathcal{T}^\mu)f(x) := \sup_{\{\epsilon_m\}} \left( \sum_{j \in \mathbb{Z}} \sum_{\epsilon_m, \epsilon_{m+1} \in I_j} |(K\chi_{\epsilon_{m+1}}^\rho * (f\mu))(x)|^\rho \right)^{1/\rho},
\]

\[
(\mathcal{V}_\rho^L \circ \mathcal{T}^\mu)f(x) := \sup_{\{\epsilon_m\}} \left( \sum_{m \in \mathbb{Z}; \epsilon_m \in I_j, \epsilon_{m+1} \in I_k} |(K\chi_{\epsilon_{m+1}}^\rho * (f\mu))(x)|^\rho \right)^{1/\rho},
\]

and, in both cases, the pointwise supremum is taken over all the sequences of positive numbers $\{\epsilon_m\}_{m \in \mathbb{Z}}$ decreasing to zero. The theorem above will be proven by showing the $L^2(\mu)$ boundedness of $\mathcal{V}_\rho \circ \mathcal{T}^\mu$ and $\mathcal{V}_\rho^S \circ \mathcal{T}^\mu$. This will be done in Subsections 5.3.1 and 5.3.4, respectively.

### 5.3.1 $L^2(\mu)$ boundedness of $\mathcal{V}_\rho^L \circ \mathcal{T}^\mu$

The $L^2(\mu)$-norm of the long variation operator $\mathcal{V}_\rho^L \circ \mathcal{T}^\mu$ can be handled by comparing it with its smoothed version $\mathcal{V}_\rho^L \circ \mathcal{T}^\mu_{\mu}$, using Corollary 5.2.2, and estimating the error terms by the short variation operator. More precisely,

\[
((\mathcal{V}_\rho^L \circ \mathcal{T}^\mu)f(x))^\rho = \sup_{\{\epsilon_m\}} \sum_{m \in \mathbb{Z}; \epsilon_m \in I_j, \epsilon_{m+1} \in I_k} |(K\chi_{\epsilon_{m+1}}^\rho * (f\mu))(x)|^\rho
\]

\[
\leq \sup_{\{\epsilon_m\}} \sum_{m \in \mathbb{Z}; \epsilon_m \in I_j, \epsilon_{m+1} \in I_k} \left( |(K\chi_{\epsilon_{m+1}}^\rho * (f\mu))(x)|^\rho + |(K\varphi_{\epsilon_{m+1}}^\rho * (f\mu))(x)|^\rho \right)
\]

\[
\leq \sup_{\{\epsilon_m\}} \sum_{m \in \mathbb{Z}; \epsilon_m \in I_j, \epsilon_{m+1} \in I_k} |(K(\chi_{\epsilon_{m+1}} - \varphi_{\epsilon_{m+1}}) * (f\mu))(x)|^\rho + ((\mathcal{V}_\rho \circ \mathcal{T}^\mu)f(x))^\rho.
\]

For simplicity, we denote by $((\mathcal{V}_\rho^L \circ \mathcal{T}^\mu_{\mu})f(x))^\rho$ the first term on the right hand side of the last inequality in (5.3.2). Notice that, given $\epsilon, \delta > 0$, we have $\chi_\epsilon - \varphi_\delta = (\chi_\epsilon - \varphi_\epsilon) - (\varphi_\delta - \varphi_\epsilon)$. Recall that, in the definition of $\varphi$ in Definition 5.0.3, we have taken $\chi_{[4,\infty)} \leq \varphi \leq \chi_{[1/4, \infty)}$ (this precise numbers are taken just for definiteness). Hence, given $t \geq 0$,

\[
\chi_R(t) - \varphi_R(t) = \chi_{[1,\infty)}(t) - \int_{1/4}^{4} \varphi_R(s)\chi_{[s,\infty)}(t) ds = \int_{1/4}^{4} \varphi_R'(s)(\chi_{[1,\infty)}(t) - \chi_{[s,\infty)}(t)) ds
\]
(χ_R − ϕ_R is a convex combination of χ_{[1,∞)} − χ_{[s,∞)} for 1/4 ≤ s ≤ 4), and thus, by Fubini’s theorem,

(K(χ_ε − ϕ_ε) ∗ (fμ))(x) = \int (χ_ε(|x − y|) − ϕ_ε(|x − y|))K(x − y)f(y) dμ(y)

= \int (χ_R(|x − y|^2/ε^2) − ϕ_R(|x − y|^2/ε^2))K(x − y)f(y) dμ(y)

= \int_{1/4}^{4} ϕ_R'(s) \int (χ_{[1,∞)}(|x − y|^2/ε^2) − χ_{[s,∞)}(|x − y|^2/ε^2))K(x − y)f(y) dμ(y) ds

= \int_{1/4}^{4} ϕ_R'(s) \int χ^\sqrt{\epsilon}(x − y)K(x − y)f(y) dμ(y) ds = \int_{1/4}^{4} ϕ_R'(s)((Kχ^\sqrt{\epsilon} ∗ (fμ))(x)) ds.

Then, by Minkowski’s integral inequality, if \{r_j\}_{j \in \mathbb{Z}} denotes a sequence such that r_j \in I_j, we have

\left(\sum_{j \in \mathbb{Z}} |(K(χ_{r_j} − ϕ_{r_j}) ∗ (fμ))(x)|^p\right)^{1/p} \leq \int_{1/4}^{4} ϕ_R'(s) \left(\sum_{j \in \mathbb{Z}} |(Kχ_{r_j}^\sqrt{\epsilon} ∗ (fμ))(x)|^p\right)^{1/p} ds.

One can easily verify that sup_{r_j \in \mathbb{Z}} \int |(Kχ_{r_j}^\sqrt{\epsilon} ∗ (fμ))(x)|^p \leq (V^S \circ T^\mu) f(x) for all s ∈ [1/4, 4] with uniform bounds. Therefore, by applying the triangle inequality and Minkowski’s integral inequality once again, we have

\| (V^C \circ T^\mu) f \|_{L^2(\mu)} \leq 2 \sup_{\{r_j \in \mathbb{Z}\}} \int |(Kχ_{r_j}^\sqrt{\epsilon} ∗ (fμ))(x)|^p \leq \int_{1/4}^{4} ϕ_R'(s) \left(\sum_{j \in \mathbb{Z}} |(Kχ_{r_j}^\sqrt{\epsilon} ∗ (fμ))(x)|^p\right)^{1/p} ds \leq C \int_{1/4}^{4} ϕ_R'(s) \left(\sum_{j \in \mathbb{Z}} |(Kχ_{r_j}^\sqrt{\epsilon} ∗ (fμ))(x)|^p\right)^{1/p} ds.

Finally, using (5.3.2), (5.3.3), and Corollary 5.2.2,

\| (V^C \circ T^\mu) f \|_{L^2(\mu)} \leq C \| (V^S \circ T^\mu) f \|_{L^2(\mu)} + \| (V^\rho \circ T^\mu) f \|_{L^2(\mu)} \leq C \| (V^S \circ T^\mu) f \|_{L^2(\mu)} + \| f \|_{L^2(\mu)}.

Thus, to prove Theorem 5.3.1, it only remains to show the L^2(\mu) boundedness of V^S \circ T^\mu, which is done in Subsection 5.3.4.

5.3.2 Corona decomposition

Given an n-dimensional AD regular Borel measure μ on \mathbb{R}^d, let \mathcal{D} := \{ Q \in \mathcal{D}_j : j \in \mathbb{Z} \} denote the dyadic lattice associated to μ in the sense of [DS2, Chapter 3 of Part I]. In [DS1] it is shown that μ is a uniformly n-rectifiable measure if and only if μ admits a corona decomposition.
finite Borel measures

where the infimum is taken over all constants $F$ the supremum norm. Recall that, given $\mu$ to prove Theorem 5.3.1. Recall from [DS2, Definitions 3.13 and 3.19 of Part I] that $\mu$ admits a corona decomposition if, for each $\eta > 0$ and $0 < \theta < 1/100$, one can find a triple $(B, G, \text{Trs})$, where $B$ and $G$ are two subsets of $D$ (the “bad cubes” and the “good cubes”) and $\text{Trs}$ is a family of subsets $S \subset G$ (commonly called trees), which satisfy the following conditions:

(a) $D = B \cup G$ and $B \cap G = \emptyset$.

(b) $B$ satisfies a Carleson packing condition: there exists $C > 0$ such that $\sum_{Q \in B: Q \subset R} \mu(Q) \leq C\mu(R)$ for all $R \in D$.

(c) $G = \bigcup_{S \in \text{Trs}} S$, i.e., any $Q \in G$ belongs to only one $S \in \text{Trs}$.

(d) Each $S \in \text{Trs}$ is coherent. This means that each $S \in \text{Trs}$ has a unique maximal element $Q_S$ which contains all other elements of $S$ as subsets, that $Q' \in S$ as soon as $Q' \in D$ satisfies $Q \subset Q' \subset Q_S$ for some $Q \in S$, and that if $Q \in S$ then either all of the children of $Q$ lie in $S$ or none of them do.

(e) The maximal cubes $Q_S$, for $S \in \text{Trs}$, satisfy a Carleson packing condition: there exists $C > 0$ such that $\sum_{S \in \text{Trs}: Q_S \subset R} \mu(Q_S) \leq C\mu(R)$ for all $R \in D$.

(f) For each $S \in \text{Trs}$, there exists a (rotation and translation of an) $n$-dimensional Lipschitz graph $\Gamma_S$ with constant smaller than $\eta$ such that $\text{dist}(x, \Gamma_S) \leq \theta \text{diam}(Q)$ whenever $x \in 2Q$ and $Q \in S$ (one can replace “$x \in 2Q$” by “$x \in C_{\text{cor}}Q$” for any constant $C_{\text{cor}} \geq 2$ given in advance, by [DS2, Lemma 3.31 of Part I]).

5.3.3 The $\alpha$ and $\beta$ coefficients with respect to $D$

Although the $\alpha$ and $\beta$ coefficients were introduced in Subsection 3.1.2 of Chapter 3, in this chapter we will adapt them to the dyadic lattice $D$. Let $\mu$ be an $n$-dimensional AD regular Borel measure on $\mathbb{R}^d$ and let $D$ denote the dyadic lattice associated to $\mu$ in the sense of [DS2, Chapter 3 of Part I]. Given $1 \leq p < \infty$ and a cube $Q \in D$, one sets (see [DS2])

$$
\beta_{p,\mu}(Q) := \inf_L \left\{ \frac{1}{\ell(Q)^n} \int_{2Q} \left( \frac{\text{dist}(y, L)}{\ell(Q)} \right)^p d\mu(y) \right\}^{1/p},
$$

where the infimum is taken over all $n$-planes $L$ in $\mathbb{R}^d$. For $p = \infty$ one replaces the $L^p$ norm by the supremum norm. Recall that, given $F \subset \mathbb{R}^d$ a the closure of an open set, and given two finite Borel measures $\sigma, \nu$ on $\mathbb{R}^d$, one sets $\text{dist}_F(\sigma, \nu) := \sup \{ \| f d\sigma - f d\nu \| : \text{Lip}(f) \leq 1, \text{supp} f \subset F \}$. Finally, given a cube $Q \in D$, consider the closed ball $B_Q := B(z_Q, 6\sqrt{d}\ell(Q))$, where $z_Q$ denotes the center of $Q$ in the sense of [DS2, Chapter 3 of Part I]. Then one defines

$$
\alpha_{\mu}^n(Q) := \frac{1}{\ell(Q)^{n+1}} \inf_{c \geq 0, L} \text{dist}_{B_Q}(\mu, c\mathcal{H}^n_L),
$$

where the infimum is taken over all constants $c \geq 0$ and all $n$-planes $L$ in $\mathbb{R}^d$. 

5.3. $L^2$ boundedness of $\mathcal{V}_p \circ T^\mu$ for uniformly rectifiable measures $\mu$

As we said in Chapter 3, the following result characterizes the uniform rectifiability of $\mu$ in terms of the $\alpha$ and $\beta$ coefficients (see [To11], for example).

**Theorem 5.3.2.** Consider any $p \in [1,2]$. Then, the following are equivalent:

(a) $\mu$ is uniformly $n$-rectifiable.

(b) For any cube $R \in \mathcal{D}$, $\sum_{Q \in \mathcal{D} : Q \subset R} \beta_{p,\mu}(Q)^2 \mu(Q) \leq C \mu(R)$ with $C$ independent of $R$.

(c) For any cube $R \in \mathcal{D}$, $\sum_{Q \in \mathcal{D} : Q \subset R} \alpha_{p,\mu}(Q)^2 \mu(Q) \leq C \mu(R)$ with $C$ independent of $R$.

5.3.4 $L^2(\mu)$ boundedness of $\mathcal{V}_p^S \circ T^\mu$

We are going to show that the short variation operator $\mathcal{V}_p^S \circ T^\mu$ is bounded in $L^2(\mu)$. Given $f \in L^2(\mu)$ and $x \in \text{supp} \mu$, let $\{\epsilon_m \equiv \epsilon_m(x)\}_{m \in \mathbb{Z}}$ be a decreasing sequence of positive numbers (depending on $x$) such that

$$((\mathcal{V}_p^S \circ T^\mu)f(x))^2 \leq 2 \sum_{j \in \mathbb{Z}} \sum_{\epsilon_m,\epsilon_{m+1} \in I_j} |(K\chi_{\epsilon_m}^j * (f\mu))(x)|^2.$$

Given $D \in \mathcal{D}_j$ and $x \in D$, we set $S_D(x) := \{m \in \mathbb{Z} : \epsilon_m, \epsilon_{m+1} \in I_j\}$. Since $\rho \geq 2$, we have

$$\|(\mathcal{V}_p^S \circ T^\mu)f\|_{L^2(\mu)}^2 \leq \|((\mathcal{V}_p^S \circ T^\mu)f\|_{L^2(\mu)}^2 \lesssim \int \sum_{j \in \mathbb{Z}} \sum_{\epsilon_m,\epsilon_{m+1} \in I_j} |(K\chi_{\epsilon_m}^j * (f\mu))(x)|^2 \, d\mu(x)$$

$$= \sum_{j \in \mathbb{Z}} \sum_{D \in \mathcal{D}_j} \int_D \sum_{\epsilon_m,\epsilon_{m+1} \in I_j} |(K\chi_{\epsilon_m}^j * (f\mu))(x)|^2 \, d\mu(x)$$

$$= \sum_{D \in \mathcal{D}} \int_D \sum_{m \in S_D(x)} |(K\chi_{\epsilon_m}^j * (f\mu))(x)|^2 \, d\mu(x).$$

Let $\eta$ and $\theta$ be two positive numbers that will be fixed below (see the proofs of Claims 5.3.3 and 5.3.4). Consider a corona decomposition of $\mu$ with parameters $\eta$ and $\theta$ as in Subsection 5.3.2. Then, we can decompose $\mathcal{D} = \mathcal{B} \cup (\bigcup_{S \in \mathcal{Tr}_S} S)$, so

$$\|(\mathcal{V}_p^S \circ T^\mu)f\|_{L^2(\mu)}^2 \lesssim \sum_{D \in \mathcal{B}} \int_D \sum_{m \in S_D(x)} |(K\chi_{\epsilon_m}^j * (f\mu))(x)|^2 \, d\mu(x)$$

$$+ \sum_{S \in \mathcal{Tr}_S} \sum_{D \in \mathcal{S}} \int_D \sum_{m \in S_D(x)} |(K\chi_{\epsilon_m}^j * (f\mu))(x)|^2 \, d\mu(x).$$  \hspace{1cm} (5.3.4)

Since the cubes in $\mathcal{B}$ satisfy a Carleson packing condition, we can use Carleson’s embedding theorem to estimate the sum on the right hand side of (5.3.4) over the cubes in $\mathcal{B}$. More
precisely, if we set $m^{\mu}_{D} f := \mu(D)^{-1} \int_{D} f \, d\mu$, we have

$$\sum_{D \in B} \int_{D} \sum_{m \in S_{D}(x)} |(K\chi_{Q_{m+1}}^{m} \ast (f\mu))(x)|^{2} \, d\mu(x)$$

$$\leq \sum_{D \in B} \int_{D} \sum_{m \in S_{D}(x)} \left( \int_{\epsilon_{m+1} \leq |x-y| \leq \epsilon_{m}} |K(x-y)||f(y)| \, d\mu(y) \right)^{2} \, d\mu(x)$$

$$\lesssim \sum_{D \in B} \int_{D} \left( \frac{1}{\ell(D)^{n}} \int_{S_{D}} |f| \, d\mu \right)^{2} \, d\mu \approx \sum_{D \in B} \left( m^{\mu}_{S_{D}}|f| \right)^{2} \mu(D) \approx \|f\|_{L^{2}(\mu)}^{2}. \quad (5.3.5)$$

We are going to estimate now the sum on the right hand side of (5.3.4) over the cubes in $S$, for all $S \in \text{Trs}$. In order to do this, we need to introduce some notation. Given $R \in D_{j}$ for some $j \in \mathbb{Z}$, let $P(R)$ denote the cube in $D_{j-1}$ which contains $R$ (the parent of $R$), and set

$$\text{Ch}(R) := \{Q \in D_{j+1} : Q \subset R\},$$

$$V(R) := \{Q \in D_{j} : Q \cap B(y, \ell(R)) \neq \emptyset \text{ for some } y \in R\}$$

(Ch($R$) is the children of $R$, and $V(R)$ is the vicinity of $R$). Notice that $P(R)$ is a cube but Ch($R$) and V($R$) are collections of cubes. It is not hard to show that the number of cubes in Ch($R$) and V($R$) is bounded by some constant depending only on $n$ and the AD regularity constant of $\mu$. If $R \in S$ for some $S \in \text{Trs}$, we denote by Tr($R$) the set of cubes $Q \in S$ such that $Q \subset R$ (the tree of $R$). Otherwise, i.e., if $R \in B$, we set Tr($R$) := $\emptyset$. Finally, if Tr($R$) $\neq \emptyset$, let Stp($R$) denote the set of cubes $Q \in B \cup (G \setminus \text{Tr}(R))$ such that $Q \subset R$ and $P(Q) \in \text{Tr}(R)$ (the stopping cubes relative to $R$), so actually $Q \subset R$. On the contrary, if $R \in B$, we set Stp($R$) := $\{R\}$.

Fix $S \in \text{Trs}$, $D \in S$, and $x \in D$; we have to estimate the sum $\sum_{m \in S_{D}(x)} |(K\chi_{Q_{m+1}}^{m} \ast (f\mu))(x)|^{2}$. Since $m \in S_{D}(x)$, we have

$$\sum_{m \in S_{D}(x)} |(K\chi_{Q_{m+1}}^{m} \ast (f\mu))(x)|^{2} = \sum_{m \in S_{D}(x)} |(K\chi_{Q_{m+1}}^{m} \ast (\chi_{\tilde{D}} f\mu))(x)|^{2}, \quad (5.3.7)$$

where $\tilde{D} := \bigcup_{R \in V(D)} R$. Since this union of cubes is disjoint, we can decompose the function $\chi_{\tilde{D}} f$ using a Haar basis adapted to $D$, in the following manner:

$$\chi_{\tilde{D}} f = \sum_{R \in V(D)} \left( m^{R}_{\mu} f \right) \chi_{R} + \sum_{Q \in \text{Tr}(R)} \Delta_{Q} f + \sum_{Q \in \text{Stp}(R)} \tilde{\Delta}_{Q} f, \quad (5.3.8)$$

where we have set

$$\Delta_{Q} f := \sum_{U \in \text{Ch}(Q)} \chi_{U}(m^{U}_{\mu} f - m^{\mu}_{Q} f), \quad \text{and} \quad \tilde{\Delta}_{Q} f := \sum_{U \in \text{Ch}(Q)} \chi_{U}(f - m^{U}_{\mu} f) = \chi_{Q}(f - m^{\mu}_{Q} f).$$

We will split the right hand side of (5.3.7) into the different parts given by (5.3.8) and, in the following subsections, we will estimate each part separately.
5.3. $L^2$ boundedness of $\mathcal{V}_\rho \circ T^\mu$ for uniformly rectifiable measures $\mu$

### 5.3.4.1 Estimate of $\sum_{m \in S_D(x)} |\sum_{R \in V(D)} \sum_{Q \in \text{Tr}(R)} (K \chi_{\epsilon_{m+1}}^\epsilon \ast (\Delta_Q f \mu))(x)|^2$

Let $C_0 > 0$ be a constant small enough. Given $m \in S_D(x)$ set $A_m(x) := A(x, \epsilon_{m+1}, \epsilon_m)$, and given $R \in V(D)$ let

$$J^1_{m,R} := \{Q \in \text{Tr}(R) : Q \cap A_m(x) \neq \emptyset, \ell(Q) \geq C_0(\epsilon_m - \epsilon_{m+1})\},$$

$$J^2_{m,R} := \{Q \in \text{Tr}(R) : Q \cap A_m(x) \neq \emptyset, \ell(Q) \leq C_0(\epsilon_m - \epsilon_{m+1})\}.$$

For $Q \in J^1_{m,R}$, we write $|(K \chi_{\epsilon_{m+1}}^\epsilon \ast (\Delta_Q f \mu))(x)| \lesssim \ell(D)^{-n} \|\chi_{A_m(x)} \Delta_Q f\|_{L^1(\mu)}$. The following claim will be proved in Subsection 5.3.4.2 below.

**Claim 5.3.3.** The following estimate holds: $\sum_{Q \in J^1_{m,R}} \ell(Q)^{n-1/2} \lesssim \ell(D)^{n-1/2}$.

Using that $V(D)$ has finitely many elements (depending only on $n$ and the AD regularity constant of $\mu$), Cauchy-Schwarz inequality, Claim 5.3.3, and the estimate above it, we obtain

$$\sum_{m \in S_D(x)} \left| \sum_{R \in V(D)} \sum_{Q \in J^1_{m,R}} (K \chi_{\epsilon_{m+1}}^\epsilon \ast (\Delta_Q f \mu))(x) \right|^2 \lesssim \sum_{R \in V(D)} \sum_{m \in S_D(x)} \left( \sum_{Q \in J^1_{m,R}} \ell(D)^{-n} \|\chi_{A_m(x)} \Delta_Q f\|_{L^1(\mu)} \right)^2 \lesssim \sum_{R \in V(D)} \sum_{m \in S_D(x)} \left( \sum_{Q \in J^1_{m,R}} \ell(Q)^{n-1/2} \right) \left( \sum_{Q \in J^1_{m,R}} \frac{\|\chi_{A_m(x)} \Delta_Q f\|_{L^1(\mu)}^2}{\ell(D)^{2n} \ell(Q)^{n-1/2}} \right) \lesssim \sum_{R \in V(D)} \sum_{Q \in \text{Tr}(R)} \frac{\|\chi_{A_m(x)} \Delta_Q f\|_{L^1(\mu)}^2}{\ell(D)^{n+1/2} \ell(Q)^{n-1/2}} \lesssim \sum_{R \in V(D)} \sum_{Q \in \text{Tr}(R)} \left( \frac{\ell(Q)}{\ell(D)} \right)^{1/2} \frac{\|\Delta_Q f\|_{L^1(\mu)}^2}{\ell(D)^n \ell(Q)^n}$.

We deal now with the cubes $Q \in J^2_{m,R}$. Let $z_Q$ denote the center of $Q$. Since $\int \Delta_Q f \, d\mu = 0$ by construction of the Haar basis, we can decompose

$$(K \chi_{\epsilon_{m+1}}^\epsilon \ast (\Delta_Q f \mu))(x) = \int (\chi_{A_m(x)}(y)K(x-y) - \chi_{A_m(x)}(z_Q)K(x-z_Q)) \Delta_Q f(y) \, d\mu(y) = \int \chi_{A_m(x)}(y) \left(K(x-y) - K(x-z_Q)\right) \Delta_Q f(y) \, d\mu(y) + \int \left(\chi_{A_m(x)}(y) - \chi_{A_m(x)}(z_Q)\right)K(x-z_Q) \Delta_Q f(y) \, d\mu(y) =: T^{1,\mu}_{m}(\Delta_Q f)(x) + T^{2,\mu}_{m}(\Delta_Q f)(x).$$

(5.3.10)

For the first term on the right hand side of the last equality, we have the standard estimate (by assuming $C_0$ small enough, so any $Q \in J^2_{m,R}$ is far from $x$)

$$|T^{1,\mu}_{m}(\Delta_Q f)(x)| \lesssim \int_{A_m(x)} \frac{|y-z_Q|}{|x-y|^{n+1}} |\Delta_Q f(y)| \, d\mu(y) \lesssim \frac{\ell(Q)}{\ell(D)^{n+1}} \|\chi_{A_m(x)} \Delta_Q f\|_{L^1(\mu)}.$$

(5.3.9)
Therefore, using this estimate and Cauchy-Schwarz inequality,

$$
\sum_{m \in S_D(x)} \left| \sum_{R \in V(D)} \sum_{Q \in J^2_{m,R}} T^{1,\mu}_m(\Delta_Q f)(x) \right|^2 \lesssim \sum_{R \in V(D)} \sum_{m \in S_D(x)} \left( \sum_{Q \in J^2_{m,R}} \frac{\ell(Q)}{\ell(D)} \| \chi_{A_m(x)} \Delta_Q f \|_{L^1(\mu)} \right)^2
$$

$$
\lesssim \sum_{R \in V(D)} \left( \sum_{Q \in \text{Tr}(R)} \frac{\ell(Q)}{\ell(D)^{n+1}} \sum_{m \in S_D(x)} \| \chi_{A_m(x)} \Delta_Q f \|_{L^1(\mu)} \right)^2
$$

$$
\lesssim \sum_{R \in V(D)} \left( \sum_{Q \in \text{Tr}(R)} \frac{\ell(Q)^{n+1}}{\ell(D)^{n+1}} \left( \sum_{Q \in \text{Tr}(R)} \frac{\| \Delta_Q f \|_{L^1(\mu)}}{\ell(Q)^n \ell(D)^n} \right) \right).$$

Since $\ell(R) = \ell(D)$ for all $R \in V(D)$, we have $\| Q \in \text{Tr}(R) \left( \frac{\ell(Q)}{\ell(D)} \right)^{n+1} \leq \sum_{Q \in D: Q \in R} \left( \frac{\ell(Q)}{\ell(D)} \right)^{n+1} \lesssim 1$. Thus, we conclude

$$
\sum_{m \in S_D(x)} \left| \sum_{R \in V(D)} \sum_{Q \in J^2_{m,R}} T^{1,\mu}_m(\Delta_Q f)(x) \right|^2 \lesssim \sum_{R \in V(D)} \sum_{Q \in \text{Tr}(R)} \left( \frac{\ell(Q)}{\ell(D)} \frac{\| \Delta_Q f \|_{L^1(\mu)}}{\ell(D)^n} \right)^{1/2} \| \Delta_Q f \|_{L^1(\mu)}^{1/2}. \tag{5.3.11}
$$

We deal now with the second term on the right hand side of (5.3.10). Given $Q \in J^2_{m,R}$, since $\text{supp}(\Delta_Q f) \subset Q$, if $Q \subset A_m(x)$ or $Q \subset (A_m(x))^c$ then we obviously have $\chi_{A_m(x)}(y) - \chi_{A_m(x)}(z_Q) = 0$ for all $y \in \text{supp}(\Delta_Q f)$. Therefore, in order to estimate the sum of $T^{2,\mu}_m(\Delta_Q f)(x)$ over all $Q \in J^2_{m,R}$, we can replace $J^2_{m,R}$ by

$$
J^3_{m,R} := \{ Q \in \text{Tr}(R) : Q \cap A_m(x) \neq \emptyset, Q \cap (A_m(x))^c \neq \emptyset, \ell(Q) \leq C_0(\epsilon_m - \epsilon_{m+1}) \}.
$$

For $m \in S_D(x)$ and $Q \in J^3_{m,R}$, we will use the estimate $| T^{2,\mu}_m(\Delta_Q f)(x) | \lesssim \ell(D)^{-n} \| \Delta_Q f \|_{L^1(\mu)}$.

Claim 5.3.4. The following holds: $\sum_{Q \in J^3_{m,R}} \ell(Q)^{n-1/2} \lesssim \ell(D)^{-n} \| \Delta_Q f \|_{L^1(\mu)}$.

Hence, using that $V(D)$ has finitely many terms, Cauchy-Schwarz inequality, assuming Claim 5.3.4 (see Subsection 5.3.4.2), and using the estimate above it, we deduce

$$
\sum_{m \in S_D(x)} \left| \sum_{R \in V(D)} \sum_{Q \in J^3_{m,R}} T^{2,\mu}_m(\Delta_Q f)(x) \right|^2 \lesssim \sum_{R \in V(D)} \sum_{m \in S_D(x)} \left( \sum_{Q \in J^3_{m,R}} \frac{\ell(Q)^{1/2-n}}{\ell(D)^{n+1/2}} \| \Delta_Q f \|_{L^1(\mu)}^2 \right)
$$

$$
\lesssim \sum_{R \in V(D)} \sum_{m \in S_D(x)} \left( \sum_{Q \in J^3_{m,R}} \frac{\ell(Q)^{1/2-n}}{\ell(D)^{n+1/2}} \| \Delta_Q f \|_{L^1(\mu)}^2 \right)^{1/2} \sum_{Q \in J^3_{m,R}} \frac{\ell(Q)^{1/2-n}}{\ell(D)^{n+1/2}} \| \Delta_Q f \|_{L^1(\mu)} \left( \frac{\epsilon_m - \epsilon_{m+1}}{\ell(D)^{n+1/2}} \right)^{1/2}
$$

$$
\lesssim \sum_{R \in V(D)} \sum_{Q \in \text{Tr}(R)} \frac{\ell(Q)}{\ell(D)^{n+1/2}} \| \Delta_Q f \|_{L^1(\mu)} \sum_{m \in S_D(x): A_m(x) \cap Q \neq \emptyset, \ell(Q) \leq C_0(\epsilon_m - \epsilon_{m+1})} \left( \frac{\epsilon_m - \epsilon_{m+1}}{\ell(D)^{n+1/2}} \right)^{1/2}.
$$
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The sum over $m$ on the right hand side of the last inequality can easily bounded by some constant depending on $C_0$, thus we finally obtain

$$\sum_{m \in S_D(x)} \left| \sum_{R \in V(D)} \sum_{Q \in \mathcal{P}_m^R} T_{m}^{2,\mu}(\Delta_Q f)(x) \right|^2 \lesssim \sum_{R \in V(D)} \sum_{Q \in \text{Tr}(R)} \left( \frac{\ell(Q)}{\ell(D)} \right)^{1/2} \|\Delta_Q f\|_{L^1(\mu)}^2 \times \frac{1}{\ell(Q)^n \ell(D)^n}. \quad (5.3.12)$$

Finally, combining (5.3.9), (5.3.10), (5.3.11), and (5.3.12), we conclude

$$\sum_{m \in S_D(x)} \left| \sum_{R \in V(D)} \sum_{Q \in \text{Tr}(R)} (K_{\chi_{_{\text{atoms} + 1}}^m} \ast (\Delta_Q f \ast \mu))(x) \right|^2 \lesssim \sum_{R \in V(D)} \sum_{Q \in \text{Tr}(R)} \left( \frac{\ell(Q)}{\ell(D)} \right)^{1/2} \|\Delta_Q f\|_{L^1(\mu)}^2 \times \frac{1}{\ell(Q)^n \ell(D)^n}, \quad (5.3.13)$$

Since $\|\Delta_Q f\|_{L^1(\mu)} \lesssim \|\Delta_Q f\|_{L^2(\mu)} \ell(Q)^{n/2}$ by Hölder’s inequality, since $V(D)$ has finitely many terms, and since $\ell(R) = \ell(D)$ for all $R \in V(D)$, we obtain

$$\sum_{S \in \text{Trs}} \sum_{D \in S} \sum_{m \in S_D(x)} \left| \sum_{R \in V(D)} \sum_{Q \in \text{Tr}(R)} (K_{\chi_{_{\text{atoms} + 1}}^m} \ast (\Delta_Q f \ast \mu))(x) \right|^2 d\mu(x)$$

$$\lesssim \sum_{S \in \text{Trs}} \sum_{D \in S} \sum_{R \in V(D)} \sum_{Q \in \text{Tr}(R)} \left( \frac{\ell(Q)}{\ell(D)} \right)^{1/2} \|\Delta_Q f\|_{L^2(\mu)}^2 \quad (5.3.14)$$

$$\leq \sum_{S \in \text{Trs}} \sum_{Q \in S} \sum_{R \in D : R \supset Q} \sum_{D \in V(R)} \left( \frac{\ell(Q)}{\ell(R)} \right)^{1/2} \|\Delta_Q f\|_{L^2(\mu)}^2$$

$$\lesssim \sum_{S \in \text{Trs}} \sum_{Q \in S} \|\Delta_Q f\|_{L^2(\mu)}^2 \leq \sum_{Q \in D} \|\Delta_Q f\|_{L^2(\mu)}^2 \leq \|f\|_{L^2(\mu)}^2.$$

This last estimate will be used in combination with (5.3.7) and (5.3.8) to bound the last term on the right hand side of (5.3.4). For proving (5.3.14), it only remains to show Claims 5.3.3 and 5.3.4.

5.3.4.2 Proof of Claims 5.3.3 and 5.3.4

Both claims will follow from the following lemma:

Lemma 5.3.5. Let $C_n > 0$ be some constant depending only on $n$, $d$, and the AD regularity constant of $\mu$, and consider $x \in D \in \mathcal{D}_j$ for some $j \in \mathbb{Z}$. Let $\epsilon \in [2^{-j-1}, 2^{-j})$. Given $k \geq j$ and $R \in V(D)$, set

$$\Lambda_k := \{Q \in \text{Tr}(R) \cap \mathcal{D}_k : Q \subset A(x, \epsilon - C_n 2^{-k}, \epsilon + C_n 2^{-k})\}.$$

Then, $\mu(\bigcup_{Q \in \Lambda_k} Q) \lesssim 2^{-k} \ell(D)^{n-1} \approx 2^{-k-j(n-1)}$. 

Proof. First of all, we can assume \(k \gg j\) (otherwise, the claim follows easily using the AD regularity of \(\mu\)), thus we may assume that \(\text{dist}(x, Q) \geq \frac{3}{4}\varepsilon\). For simplicity, set \(S \equiv \text{Tr}(R)\). By the corona decomposition of \(\mu\), there exists a (rotation and translation of an) \(n\)-dimensional Lipschitz graph \(\Gamma_S\) with \(\text{Lip}(\Gamma_S) \leq \eta\) such that \(\text{dist}(y, \Gamma_S) \leq \theta \text{diam}(Q)\) whenever \(y \in C_{\text{cor}}Q\) and \(Q \in S\), for some given constant \(C_{\text{cor}} \geq 2\) (see property (f) of the corona decomposition in Subsection 5.3.2). Since \(x \in D\) and \(R \in V(D)\), we have \(x \in C_{\text{cor}}Q\) assuming \(C_{\text{cor}}\) big enough, and so \(\text{dist}(x, \Gamma_S) \leq \theta \text{diam}(Q)\). Hence, if \(\eta\) and \(\theta\) are small enough, one can easily modify \(\Gamma_S\) inside \(B(x, \frac{1}{4}\varepsilon)\) to obtain a Lipschitz graph \(\Gamma^x_S\) such that \(x \in \Gamma^x_S\), and moreover

\[
\text{Lip}(\Gamma^x_S) \leq \eta' \text{ for some } \eta' < 1, \quad \text{and } \Gamma^x_S \setminus B(x, \varepsilon/4) = \Gamma_S \setminus B(x, \varepsilon/4).
\]

Using that \(\text{dist}(x, Q) \geq \frac{3}{4}\varepsilon\) for all \(Q \in \Lambda_k\), that \(\text{dist}(y, \Gamma_S) \leq \theta \text{diam}(Q)\) for all \(y \in CQ\), and the last part of (5.3.15), we deduce that \(\text{dist}(y, \Gamma^x_S) \leq \theta \text{diam}(Q)\) for all \(Q \in \Lambda_k\) and all \(y \in CQ\). Let \(z_Q\) denote the center of \(Q\), then \(\text{dist}(z_Q, \Gamma^x_S) \leq \theta \text{diam}(Q)\), so \(B(z_Q, \theta \text{diam}(Q)) \cap \Gamma^x_S \neq \emptyset\), which in turn yields \(\mathcal{H}^n(\Gamma^x_S \cap B(z_Q, 2\theta \text{diam}(Q))) \geq (\theta \text{diam}(Q))^n\). Therefore, since \(\{B(z_Q, 2\theta \text{diam}(Q))\}_{Q \in \Lambda_k}\) is a family with finite overlap bounded by some constant depending only on \(n, \theta\), and the AD regularity constant of \(\mu\), we have

\[
\mu\left(\bigcup_{Q \in \Lambda_k} Q\right) = \sum_{Q \in \Lambda_k} \mu(Q) \approx \sum_{Q \in \Lambda_k} \ell(Q)^n \lesssim \theta^{-n} \sum_{Q \in \Lambda_k} \mathcal{H}^n(\Gamma^x_S \cap B(z_Q, 2\theta \text{diam}(Q)))
\]

\[
\lesssim \theta^{-n} \mathcal{H}^n(\Gamma^x_S) \left(\bigcup_{Q \in \Lambda_k} B(z_Q, 2\theta \text{diam}(Q))\right)
\]

\[
\lesssim \theta^{-n} \mathcal{H}^n(\Gamma^x_S) \left(A(x, \varepsilon - C\eta 2^{-k}, \varepsilon + C\eta 2^{-k})\right) \lesssim \theta^{-n} 2^{-k-j(n-1)}
\]

where we used Lemma 4.1.2 and that \(\varepsilon \approx 2^{-j}\) in the last inequality. The lemma is proved, because \(\theta\) is a fixed constant from the corona decomposition. \(\square\)

Proof of Claim 5.3.3. Recall that \(J^R_m := \{Q \in \text{Tr}(R) : Q \cap A_m(x) \neq \emptyset, \ell(Q) \geq C_{\text{cor}}(\epsilon_m - \epsilon_{m+1})\}\), where \(R \in V(D)\) and \(D \in \mathcal{D}_j\). We have to check that \(\sum_{Q \in J^R_m} \ell(Q)^{n-1/2} \lesssim \ell(D)^{n-1/2}\).

We will split the sum into different scales and we will apply Lemma 5.3.5 at each scale.

Given \(i \in \mathbb{Z}\) such that \(2^{-i} \geq C_{\text{cor}}(\epsilon_m - \epsilon_{m+1})\), the number of cubes \(Q \in \mathcal{D}\) such that \(\ell(Q) = 2^{-i}\), \(Q \subset R\), and \(Q \cap A_m(x) \neq \emptyset\) is bounded by \(C\ell(R)^{n-1/2}(2^{-i(n-1)} \approx 2^{-i(n-1)} + i(n-1)}\), since for all these cubes, \(Q \subset A(x, \epsilon_{m+1} + C2^{-i}, \epsilon_m + C2^{-i}) \subset A(x, \epsilon_m - C2^{-i+1}, \epsilon_m + C2^{-i+1})\) for some constant \(C > 0\) big enough, and by Lemma 5.3.5, \(\mu(\bigcup_{Q \in J^R_m \cap \mathcal{D}_i} Q) \lesssim 2^{-i} \ell(D)^{n-1}\). Therefore,

\[
\sum_{Q \in J^R_m} \ell(Q)^{n-1/2} = \sum_{i \in \mathbb{Z}, i \geq j} 2^{i/2} \sum_{Q \in J^R_m \cap \mathcal{D}_i} \ell(Q)^n \lesssim \sum_{i \in \mathbb{Z}, i \geq j} 2^{i/2} 2^{-i} \ell(D)^{n-1}
\]

\[
\approx 2^{-j/2} \ell(D)^{n-1} = \ell(D)^{n-1/2}.
\]

\(\square\)
5.3. $L^2$ boundedness of $\mathcal{V}_\rho \circ T^\mu$ for uniformly rectifiable measures $\mu$

**Proof of Claim 5.3.4.** Recall that $J^3_m := \{ Q \in \text{Tr}(R) : Q \cap A_m(x) \neq \emptyset, Q \cap (A_m(x))^c \neq \emptyset, \ell(Q) \leq C_0(\epsilon_m - \epsilon_{m+1}) \}$, where $R \in V(D)$ and $D \in \mathcal{D}_j$. We have to check that

$$\sum_{Q \in J^3_m} \ell(Q)^{n-1/2} \lesssim \ell(D)^{n-1}(\epsilon_m - \epsilon_{m+1})^{1/2}. $$

As before, we will split the sum in the different scales and we will apply Lemma 5.3.5 at each scale. Given $i \in \mathbb{Z}$ such that $2^{-i} \leq C_0(\epsilon_m - \epsilon_{m+1})$, for any $Q \in J^3_m \cap \mathcal{D}_i$ we have $Q \subset A(x, \epsilon_{m+1} - C2^{-i}, \epsilon_m + C2^{-i}) \cup A(x, \epsilon_m - C2^{-i}, \epsilon_m + C2^{-i})$ for some constant $C > 0$ big enough, by Lemma 5.3.5 applied to both annuli we have $\mu(\bigcup_{Q \in J^3_m \cap \mathcal{D}_i} Q) \lesssim 2^{-i}\ell(D)^{n-1}$. Therefore,

$$\sum_{Q \in J^3_m} \ell(Q)^{n-1/2} = \sum_{i \in \mathbb{Z}, i \geq -\log_2(C_0(\epsilon_m - \epsilon_{m+1}))} 2^{i/2} \sum_{Q \in J^3_m \cap \mathcal{D}_i} \ell(Q)^n \lesssim \sum_{i \in \mathbb{Z}, i \geq -\log_2(C_0(\epsilon_m - \epsilon_{m+1}))} 2^{-i/2}\ell(D)^{n-1} \approx (\epsilon_m - \epsilon_{m+1})^{1/2}\ell(D)^{n-1}. $$

\[\square\]

**5.3.4.3 Estimate of $\sum_{m \in S_D(x)} |\sum_{R \in V(D)} \sum_{Q \in \text{Stp}(R)} (K\chi^{\epsilon_{m+1}}_{\epsilon_m} \ast (\check{\Delta}_Q f \mu))(x)|^2$**

Given $R \in V(D)$, consider a cube $Q \in \text{Stp}(R)$. If $\text{Tr}(R) \neq \emptyset$, then $Q \in \mathcal{B} \cup (\mathcal{G} \setminus \text{Tr}(R))$, $Q \subset R$ and $P(Q) \in \text{Tr}(R)$ (in particular, $Q \nsubseteq R$). Take $S \in \text{Trs}$ such that $R \in S$. By property (f) of the corona decomposition (see Subsection 5.3.2), we have dist$(y, \Gamma_S) \leq \theta \text{diam}(P(Q))$ for all $y \in C_{\text{cor}}P(Q)$. Hence, dist$(y, \Gamma_S) \leq C\theta \text{diam}(Q)$ for all $y \in C_{\text{cor}}Q$, for some constant $C$ depending only on the AD regularity constant of $\mu$. On the other hand, if $\text{Tr}(R) = \emptyset$ we have set $\text{Stp}(R) = \{ R \}$. In this case, we have $R \in \mathcal{B}$. Take $S$ such that $D \in S$. Since $R \in V(D)$, we have $R \subset C_{\text{cor}}D$ if $C_{\text{cor}}$ is big enough, and thus dist$(y, \Gamma_S) \leq C\theta \text{diam}(R)$ for all $y \in C^*R$, where $C$ is as above and $C^*$ depends on $C_{\text{cor}}$.

Taking into account the comments above, one can prove the following claims using similar arguments to the one given in the proof of Claims 5.3.3 and 5.3.4.

**Claim 5.3.6.** Let $x \in D \in \mathcal{D}$, $R \in V(D)$, and $m \in S_D(x)$. If we set $J^1_m := \{ Q \in \text{Stp}(R) : Q \cap A_m(x) \neq \emptyset, \ell(Q) \geq C_0(\epsilon_m - \epsilon_{m+1}) \}$, then $\sum_{Q \in J^1_m} \ell(Q)^{n-1/2} \lesssim \ell(D)^{n-1/2}$.

**Claim 5.3.7.** Let $x \in D \in \mathcal{D}$, $R \in V(D)$, and $m \in S_D(x)$. If we set $J^3_m := \{ Q \in \text{Stp}(R) : Q \cap A_m(x) \neq \emptyset, Q \cap (A_m(x))^c \neq \emptyset, \ell(Q) \leq C_0(\epsilon_m - \epsilon_{m+1}) \}$, then $\sum_{Q \in J^3_m} \ell(Q)^{n-1/2} \lesssim \ell(D)^{n-1}(\epsilon_m - \epsilon_{m+1})^{1/2}$.

To obtain estimate (5.3.13), the only property of $\Delta_Q f$ that we used was that $\int \Delta_Q f \, d\mu = 0$. The functions $\tilde{\Delta}_Q f$ also have vanishing integral. Thus, if we replace $\text{Tr}(R)$ by $\text{Stp}(R)$,
Claims 5.3.3 and 5.3.4 by Claims 5.3.6 and 5.3.7, and \( \Delta_Qf \) by \( \tilde{\Delta}_Qf \), the same arguments that gave us (5.3.13) yield the following estimate:

\[
\sum_{m \in S_D(x)} \left| \sum_{R \in V(D)} \sum_{Q \in \text{Stp}(R)} (K\chi_{e_{m+1}} \ast (\tilde{\Delta}_Q f \mu))(x) \right|^2 \lesssim \sum_{R \in V(D)} \sum_{Q \in \text{Stp}(R)} \left( \frac{\ell(Q)}{\ell(D)} \right)^{1/2} \frac{\|\tilde{\Delta}_Qf\|_{L^1(\mu)}}{\ell(Q)^n \ell(D)^n}.
\]

(5.3.17)

We can easily estimate

\[
\|\tilde{\Delta}_Qf\|_{L^1(\mu)}^2 \lesssim \left( \int_Q |f - m_Q f| \, d\mu \right)^2 \ell(Q)^{-n} \lesssim \left( \int_Q |f| \, d\mu + (m_Q^\mu |f|) \mu(Q) \right)^2 \mu(Q)^{-1} \lesssim (m_Q^\mu |f|)^2 \mu(Q).
\]

Notice that, by the definition of \( \text{Stp}(R) \) and since the corona decomposition is coherent (property (d)), any \( Q \in \text{Stp}(R) \) is actually a maximal cube \( Q_S \) of some \( S \in \text{Trs} \) or \( Q \in \mathcal{B} \) (and in this case \( \text{Tr}(R) \) is empty). Hence, if we integrate (5.3.17) in \( D \), we sum over all \( D \in S \in \text{Trs} \), and we change the order of summation, we get

\[
\sum_{S \in \text{Trs}} \sum_{D \in S} \int_D \sum_{m \in S_D(x)} \left| \sum_{R \in V(D)} \sum_{Q \in \text{Stp}(R)} (K\chi_{e_{m+1}} \ast (\tilde{\Delta}_Q f \mu))(x) \right|^2 \, d\mu(x)
\]

\[
\lesssim \sum_{S \in \text{Trs}} \sum_{D \in S} \sum_{R \in V(D)} \sum_{Q \in \text{Stp}(R)} \left( \frac{\ell(Q)}{\ell(D)} \right)^{1/2} \frac{\|\tilde{\Delta}_Qf\|_{L^1(\mu)}}{\ell(Q)^n \ell(D)^n}
\]

\[
\lesssim \sum_{D \in \mathcal{D}} \sum_{R \in V(D)} \sum_{S \in \text{Trs} : Q_S \subset R} \left( \frac{\ell(Q_S)}{\ell(D)} \right)^{1/2} (m_Q^\mu |f|)^2 \mu(Q_S)
\]

\[
+ \sum_{D \in \mathcal{D}} \sum_{R \in V(D)} \sum_{Q \in \mathcal{B} : Q \subset R} \left( \frac{\ell(Q)}{\ell(D)} \right)^{1/2} (m_Q^\mu |f|)^2 \mu(Q)
\]

\[
= \sum_{S \in \text{Trs}} \sum_{R \in \mathcal{D} : R \subset Q_S} \sum_{D \in V(R)} \left( \frac{\ell(Q_S)}{\ell(R)} \right)^{1/2} (m_Q^\mu |f|)^2 \mu(Q_S)
\]

\[
+ \sum_{Q \in \mathcal{B}} \sum_{R \in \mathcal{D} : R \subset Q} \sum_{D \in V(R)} \left( \frac{\ell(Q)}{\ell(R)} \right)^{1/2} (m_Q^\mu |f|)^2 \mu(Q).
\]

(5.3.18)

Finally, using that \( V(R) \) has finitely many terms, and that the maximal cubes \( Q_S \) with \( S \in \text{Trs} \) and the cubes \( Q \in \mathcal{B} \) satisfy a Carleson packing condition (so we can apply Carleson’s
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embedding theorem), (5.3.18) yields

$$\sum_{S \in \text{Trs}} \sum_{D \in \mathcal{D}} \sum_{m \in S_D(x)} \sum_{R \in V(D)} \sum_{Q \in \text{Stp}(R)} (K \chi_{\ell_{m+1}}^m * (\tilde{\Delta}_Q f \mu))(x) \left| d\mu(x) \right|^2 \leq \sum_{S \in \text{Trs}} \sum_{D \in \mathcal{D}} \sum_{m \in S_D(x)} (m_Q^m |f|)^2 \mu(Q) \sum_{N \in S_D} \sum_{R \in \mathcal{D} : R \supset Q} \ell(Q)^{1/2} \ell(R)^{1/2} \sum_{Q \in \mathcal{B}} (m_Q^m |f|)^2 \mu(Q) \leq \|f\|_{L^2(\mu)}^2.$$  

(5.3.19)

5.3.4.4 Estimate of $\sum_{m \in S_D(x)} \sum_{R \in V(D)} (K \chi_{\ell_{m+1}}^m * (m_R^m f \mu))(x) \left| d\mu(x) \right|^2$

We will need the following auxiliary proposition, whose proof is given despite we think it is already known.

**Proposition 5.3.8.** Given $D \in \mathcal{D}$ and $f \in L^2(\mu)$, set $a_D(f) := \sum_{R \in V(D)} |m_R^m f - m_R^m f|$. Then, there exists $C > 0$ depending only on $n$ and the AD regularity constant of $\mu$ such that

$$\sum_{D \in \mathcal{D}} (a_D(f))^2 \mu(D) \leq C \|f\|_{L^2(\mu)}^2.$$  

**Proof.** By subtracting a constant if it is necessary, we can assume that $f$ has mean zero (notice that $a_D(f) = 0$ if $f$ is constant). Consider the representation of $f$ with respect to the Haar basis related to $D$, that is $f = \sum_{Q \in \mathcal{D}} \Delta_Q f$. For $m \in \mathbb{Z}$, we define the function $u_m = \sum_{Q \in \mathcal{D}_m} \Delta_Q f$, so $f = \sum_{m \in \mathbb{Z}} u_m$ and the equality holds in $L^2(\mu)$. Given $j \in \mathbb{Z}$, define the operator

$$S_j(f) := \left( \sum_{D \in \mathcal{D}_j} (a_D(f))^2 \chi_D \right)^{1/2}.$$  

We will prove that there exists a sequence $\{\sigma(k)\}_{k \in \mathbb{Z}}$ such that

$$\sum_{k \in \mathbb{Z}} \sigma(k) \leq C < \infty \quad \text{and} \quad \|S_j(u_m)\|_{L^2(\mu)} \leq \sigma(|m - j|) \|u_m\|_{L^2(\mu)}.$$  

(5.3.20)

Assume for the moment that (5.3.20) holds. Obviously, $S_j$ is a sublinear operator for all $j \in \mathbb{Z}$ (because of the subadditivity of $a_D$ and the triangle inequality). Then, using Cauchy-
Schwarz inequality and the orthogonality of the \(u_m\)'s (inherited from the Haar basis),
\[
\sum_{D \in \mathcal{D}} (a_D(f))^2 \mu(D) = \sum_{j \in \mathbb{Z}} \int_{D \in \mathcal{D}_j} (a_D(f))^2 \chi_D \, d\mu = \sum_{j \in \mathbb{Z}} \|S_j(f)\|_{L^2(\mu)}^2
\]
\[
= \sum_{j \in \mathbb{Z}} \left\| S_j \left( \sum_{m \in \mathbb{Z}} u_m \right) \right\|_{L^2(\mu)}^2 \leq \sum_{j \in \mathbb{Z}} \left( \sum_{m \in \mathbb{Z}} \|S_j(u_m)\|_{L^2(\mu)}^2 \right)^2 \leq \sum_{j \in \mathbb{Z}} \left( \sum_{m \in \mathbb{Z}} \|u_m\|_{L^2(\mu)}^2 \right) \sum_{j \in \mathbb{Z}} \sigma(|m - j|)
\]
\[
\lesssim \sum_{m \in \mathbb{Z}} \|u_m\|_{L^2(\mu)}^2 = \|f\|_{L^2(\mu)}^2,
\]
and the Proposition is proved, except for (5.3.20). Let us verify it. By definition,
\[
\|S_j(u_m)\|_{L^2(\mu)}^2 = \sum_{D \in \mathcal{D}_j} \left( \sum_{R \in \mathcal{V}(D)} \int_{Q \in \mathcal{D}_m} \int \Delta_Q f \left( \frac{\chi_R}{\mu(R)} - \frac{\chi_D}{\mu(D)} \right) \, d\mu \right)^2 \mu(D). \tag{5.3.21}
\]
Assume first that \(m \geq j\). If \(D \in \mathcal{D}_j, R \in \mathcal{V}(D)\), and \(Q \in \mathcal{D}_m\), then either \(Q \cap R = \emptyset\) or \(Q \subset R\). In both cases, since \(\Delta_Q f\) has mean zero and is supported in \(Q\), we have \(\int \Delta_Q f \chi_R \, d\mu = 0\). Thus, the right hand side of (5.3.21) vanishes (obviously \(D \in \mathcal{V}(D)\)), and (5.3.20) follows.

Assume now that \(m < j\). Set \(\tilde{D} := \bigcup_{R \in \mathcal{V}(D)} R\). Recall that \(\Delta_Q f := \sum_{U \in \text{Ch}(Q)} \chi_U (m_U^f - m_Q^f)\), so \(\Delta_Q f\) is constant in each \(U \in \text{Ch}(Q)\). Hence, if for some \(U \in \text{Ch}(Q)\) we have \(\tilde{D} \subset U\) or \(\tilde{D} \subset \text{supp} \mu \setminus U\), then \((R \cup D) \subset U\) or \((R \cup D) \cap U = \emptyset\) for all \(R \in \mathcal{V}(D)\), and so
\[
\int \chi_U (m_U^f - m_Q^f) \left( \frac{\chi_R}{\mu(R)} - \frac{\chi_D}{\mu(D)} \right) \, d\mu = (m_U^f - m_Q^f) \int_U \left( \frac{\chi_R}{\mu(R)} - \frac{\chi_D}{\mu(D)} \right) \, d\mu = 0
\]
for all \(R \in \mathcal{V}(D)\). Therefore, if we set \(m_{U,Q}^f := (m_U^f - m_Q^f)\), using that \(\mathcal{V}(D)\) has finitely many terms and that \(\int |\mu(R)^{-1} \chi_R - \mu(D)^{-1} \chi_D| \, d\mu \leq 2\) for all \(R \in \mathcal{V}(D)\), we deduce from (5.3.21) that
\[
\|S_j(u_m)\|_{L^2(\mu)}^2 = \sum_{D \in \mathcal{D}_j} \left( \sum_{R \in \mathcal{V}(D)} \sum_{Q \in \mathcal{D}_m} \sum_{U \in \text{Ch}(Q)} \chi_U m_{U,Q}^f \left( \frac{\chi_R}{\mu(R)} - \frac{\chi_D}{\mu(D)} \right) \, d\mu \right)^2 \mu(D)
\]
\[
\leq 4 \sum_{D \in \mathcal{D}_j} \left( \sum_{Q \in \mathcal{D}_m} \sum_{U \in \text{Ch}(Q)} \left| m_{U,Q}^f \right| \right)^2 \mu(D) = 4 \sum_{D \in \mathcal{D}_j} \left( \sum_{U \in \mathcal{D}_{m+1} \setminus \text{supp} \mu, \, \text{supp} \mu \cap U \neq \emptyset} \left| m_{U,P(U)}^f \right| \right)^2 \mu(D). \tag{5.3.22}
\]
It is not hard to show that, since \(m < j\) and \(D \in \mathcal{D}_j\), the number of cubes \(U \in \mathcal{D}_{m+1}\) such that \(\tilde{D} \cap U \neq \emptyset\) and \(\tilde{D} \cap U^c \neq \emptyset\) is bounded by some constant depending only on \(n\) and...
the AD regularity constant of $\mu$ (in particular, this constant does not depend on the precise value of $m$). Hence,

$$
\sum_{D \in D_j} \left( \sum_{U \in D_{m+1}; \text{Dr}_U \neq \emptyset, \text{Dr}_U \cap e \neq \emptyset} |m_{U,P(U)}f|^2 \right)^2 \mu(D) \lesssim \sum_{D \in D_j} \sum_{U \in D_{m+1}; \text{Dr}_U \neq \emptyset, \text{Dr}_U \cap e \neq \emptyset} |m_{U,P(U)}f|^2 \mu(D)
$$

$$
= \sum_{U \in D_{m+1}} |m_{U,P(U)}f|^2 \sum_{D \in D_j; \text{Dr}_U \neq \emptyset, \text{Dr}_U \cap e \neq \emptyset} \mu(D) = \sum_{U \in D_{m+1}} |m_{U,P(U)}f|^2 \mu(D) \lesssim \sum_{D \in D_j; \text{Dr}_U \neq \emptyset, \text{Dr}_U \cap e \neq \emptyset} \mu(D).
$$

(5.3.23)

Fix $U \in D_{m+1}$. Recall that $\tilde{D} := \bigcup_{R \in V(D)} R$, so $\text{diam}(\tilde{D}) \approx \text{diam}(D)$. Hence, there exists a constant $\tau_0 > 0$ such that

$$
\bigcup_{D \in D_j; \text{Dr}_U \neq \emptyset, \text{Dr}_U \cap e \neq \emptyset} D \subset \{ x \in U : \text{dist}(x, \text{supp}\mu \setminus U) \leq \tau_0 \ell(D) \}
$$

$$
\cup \{ x \in \text{supp}\mu \setminus U : \text{dist}(x, U) \leq \tau_0 \ell(D) \}
$$

$$
= \{ x \in U : \text{dist}(x, \text{supp}\mu \setminus U) \leq \tau_0 2^{m-j+1} \ell(U) \}
$$

$$
\cup \{ x \in \text{supp}\mu \setminus U : \text{dist}(x, U) \leq \tau_0 2^{m-j+1} \ell(U) \}.
$$

If $m \ll j$, then $\tau := \tau_0 2^{m-j+1} < 1$, so we can apply the small boundaries condition (5.2.1) to obtain $\mu\left( \bigcup_{D \in D_j; \text{Dr}_U \neq \emptyset, \text{Dr}_U \cap e \neq \emptyset} D \right) \lesssim C_{1,C} 2^{-mn}$. On the contrary, if $m$ and $j$ are not far from one another, then $\tau^{1/C} \approx 1$, so $\mu\left( \bigcup_{D \in D_j; \text{Dr}_U \neq \emptyset, \text{Dr}_U \cap e \neq \emptyset} D \right) \leq \mu(C_1 U) \lesssim 2^{-mn} \approx \tau^{1/C} 2^{-mn}$, for some big constant $C_2 > 0$. Thus, in any case, we have seen that $\mu\left( \bigcup_{D \in D_j; \text{Dr}_U \neq \emptyset, \text{Dr}_U \cap e \neq \emptyset} D \right) \lesssim 2^{(m-j)/C} \ell(U)^n$, and combining this estimate with (5.3.23) and (5.3.22) we conclude that, for $m < j$,

$$
\|S_j(u_m)\|^2_{L^2(\mu)} \lesssim 2^{(m-j)/C} \sum_{U \in D_{m+1}} |m_{U,P(U)}f|^2 \ell(U)^n
$$

$$
\approx 2^{(m-j)/C} \int \sum_{U \in D_{m+1}} \chi_U |m_{U,P(U)}f|^2 d\mu
$$

$$
= 2^{(m-j)/C} \int \left| \sum_{Q \in D_m} \sum_{U \in \text{Ch}(Q)} \chi_U (m_{U,P(U)}f - m_{Q,P(U)}f) \right|^2 d\mu = 2^{-|m-j|/C} \|u_m\|^2_{L^2(\mu)};
$$

which gives (5.3.20) with $\sigma(k) = 2^{-\frac{|k|}{m}}$ and finishes the proof of the proposition. 

\[\square\]
Recall that, given $D \in \mathcal{D}$, we have set $\tilde{D} := \bigcup_{R \in \mathcal{V}(D)} R$. We have
\[
\sum_{m \in \mathcal{S}_D(x)} \left| \sum_{R \in \mathcal{V}(D)} (K\chi_{\epsilon_{m+1}} \ast ((m_R^\mu f)(\chi_{R\mu}))(x)) \right|^2 \lesssim \sum_{m \in \mathcal{S}_D(x)} \left| (K\chi_{\epsilon_{m+1}} \ast ((m_R^\mu f)(\chi_{D\mu}))(x)) \right|^2 \\
+ \sum_{m \in \mathcal{S}_D(x)} \left| \sum_{R \in \mathcal{V}(D)} (K\chi_{\epsilon_{m+1}} \ast ((m_R^\mu f - m_D^\mu f)(\chi_{R\mu}))(x)) \right|^2 \tag{5.3.24}
\]
We are going to estimate the two terms on the right hand side of (5.3.24) separately. For the second one, recall also that, given $m \in \mathcal{S}_D(x)$, we have also set $A_m(x) := A(x, \epsilon_{m+1}, \epsilon_m)$. We write
\[
|(K\chi_{\epsilon_{m+1}} \ast ((m_R^\mu f - m_D^\mu f)(\chi_{R\mu}))(x))| \leq |m_R^\mu f - m_D^\mu f| \int_{A_m(x)} |K(x-y)|\chi_{R}(y) \, d\mu(y) \\
\lesssim |m_R^\mu f - m_D^\mu f| \mu(A_m(x) \cap R)\ell(D)^{-\gamma}.
\]
Therefore, interchanging the order of summation,
\[
\sum_{m \in \mathcal{S}_D(x)} \left| \sum_{R \in \mathcal{V}(D)} (K\chi_{\epsilon_{m+1}} \ast ((m_R^\mu f - m_D^\mu f)(\chi_{R\mu}))(x)) \right|^2 \\
\lesssim \left( \sum_{m \in \mathcal{S}_D(x)} \sum_{R \in \mathcal{V}(D)} |m_R^\mu f - m_D^\mu f| \mu(A_m(x) \cap R)\ell(D)^{-\gamma} \right)^2 \\
\lesssim \left( \sum_{R \in \mathcal{V}(D)} |m_R^\mu f - m_D^\mu f| \frac{\mu(R)}{\ell(D)^n} \right)^2 \approx \left( \sum_{R \in \mathcal{V}(D)} |m_R^\mu f - m_D^\mu f| \right)^2 = (a_D(f))^2,
\]
where $a_D(f)$ are the coefficients introduced in Proposition 5.3.8. If we integrate on $D$ and sum over all $D \in S$ and $S \in \mathcal{V}$, we can apply Proposition 5.3.8, and we finally obtain
\[
\sum_{S \in \mathcal{S}} \sum_{D \in \mathcal{S}} \int_D \sum_{m \in \mathcal{S}_D(x)} \left| \sum_{R \in \mathcal{V}(D)} (K\chi_{\epsilon_{m+1}} \ast ((m_R^\mu f - m_D^\mu f)(\chi_{R\mu}))(x)) \right|^2 \, d\mu(x) \\
\lesssim \sum_{D \in \mathcal{D}} (a_D(f))^2 \mu(D) \lesssim \|f\|_{L^2(\mu)}^2. \tag{5.3.25}
\]
Let us estimate now the first term on the right hand side of (5.3.24). We will use almost the same arguments (and notation) as the ones of Subsection 4.1.2.3. For the reader’s convenience, we will expose the adapted arguments in detail. Let $L_D$ be a minimizing $n$-plane for $\alpha_{\mu}(D)$ and let $L_D^x$ be the $n$-plane parallel to $L_D$ which contains $x$. Given $z \in \mathbb{R}^d$, let $p_0^x$ denote the orthogonal projection onto $L_D^x$. Let $g_1, g_2 : \mathbb{R} \to [0, 1]$ be such that \( \text{supp} g_1 \subset (-2\varepsilon \ell(D), 2\varepsilon \ell(D)) \), \( \text{supp} g_2 \subset (-\ell(D)\varepsilon, \ell(D)\varepsilon)^c \), and $g_1 + g_2 = 1$, where $\varepsilon > 0$ is some fixed constant small enough. For $z \in \mathbb{R}^d$, consider the projection onto $L_D^x$ given by
\[
p^x(z) := \left( x + (p_0^x(z) - x) \frac{|z - x|}{|p_0^x(z) - x|} \right) g_2(|p_0^x(z) - x|) + p_0^x(z)g_1(|p_0^x(z) - x|). \tag{5.3.26}
\]
Then, we can easily estimate $g_{5.3}$. for all $\text{dist}(\varepsilon p_\tilde{L})$ Lipschitz constant depending only on $(1/p)$.

Let $C_\ast > 0$ be a small constant which will be fixed below. Assume that $\alpha_\mu(10D) \geq C_\ast$.

Then, we can easily estimate

$$\sum_{m \in S_D(x)} |(K \chi_{m+1}^* ((m_D^\mu f) \chi_D \mu))(x)|^2 = |m_D^\mu f|^2 \sum_{m \in S_D(x)} \left| \int_{A_m(x) \cap D} K(x - y) d\mu(y) \right|^2$$

$$\lesssim |m_D^\mu f|^2 \left( \sum_{m \in S_D(x)} \int_{A_m(x) \cap D} |K(x - y)| d\mu(y) \right)^2 \lesssim |m_D^\mu f|^2 \lesssim |m_D^\mu f|^2 \alpha_\mu(10D)^2.$$

From now on, we assume that $\alpha_\mu(10D) < C_\ast$. By assuming $C_\ast$ small enough, it is not difficult to show that then the distance between $D$ and $L_D^\ast$ is smaller than $\ell(D)/1000$. Moreover, $p^\ast$ restricted to $\{y \in A_m(x) : \text{dist}(y, L_D^\ast) \leq \ell(D)/1000\}$ is a Lipschitz function with Lipschitz constant depending only on $n, d$, and the AD regularity constant of $\mu$. Furthermore, by taking $\varepsilon$ small enough, we have

$$p^\ast(z) = x + (p_0^\ast(z) - x) \frac{|z - x|}{|p_0^\ast(z) - x|}$$

for all $z \in \{y \in \hat{D} \cap A_m(x) : \text{dist}(y, L_D^\ast) \leq \ell(D)/1000\} \subset \text{supp}\mu$.

Recall that $D \in S$ for some $S \in \text{Trs}$. Let $Q_S$ be the maximal cube of $S$, and set $\nu_\varepsilon := p_0^\ast(\chi_{Q_{400}S} \mu)$. Then, since $\text{supp}\mu \cap A_m(x) \subset \hat{D}$ by the construction of $\hat{D}$,

$$(K \chi_{m+1}^* ((m_D^\mu f) \chi_D \mu))(x) = (m_D^\mu f) \int_{A_m(x)} K(x - y) d\mu(y)$$

$$= (m_D^\mu f) \int_{A_m(x)} K(x - y) d(\mu - \nu_\varepsilon)(y) + (m_D^\mu f) \int_{A_m(x)} K(x - y) d\nu_\varepsilon(y)$$

$$=: U1_m(x) + U2_m(x).$$

Notice that $y \in A_m(x)$ if and only if $p^\ast(y) \in A_m(x)$. Since $|y - p^\ast(y)| \lesssim \text{dist}(y, L_D^\ast) \leq \text{dist}(y, L_D) + \text{dist}(x, L_D)$ for all $y \in \Gamma$,

$$|U1_m(x)| \leq |m_D^\mu f| \int_{A_m(x)} |K(x - y) - K(x - p^\ast(y))| d\mu(y) \lesssim \frac{|m_D^\mu f|}{\ell(D)^{n+1}} \int_{A_m(x)} |y - p^\ast(y)| d\mu(y) \approx \frac{|m_D^\mu f|}{\ell(D)^{n+1}} \int_{A_m(x)} (\text{dist}(y, L_D) + \text{dist}(x, L_D)) d\mu(y).$$

If $L_D^\ast$ denotes a minimizing $n$-plane for $\beta_1(D)$, one can show that $\text{dist}_H(L_D \cap B_D, L_D^\ast \cap B_D) \lesssim$
5. Variation for Riesz transforms and uniform rectifiability

\[ \alpha_\mu(D)\ell(D), \text{so dist}(y, L_D) \leq \text{dist}(y, L^1_D) + \alpha_\mu(D)\ell(D) \text{ for } y \in C_D \cap \Gamma. \]  
Therefore,

\[
\sum_{m \in \mathcal{S}_D(x)} |U_{1m}(x)|^2 \lesssim \left( \frac{m_D^n f}{\ell(D)^{n+1}} \sum_{m \in \mathcal{S}_D(x)} \int_{A_m(x)} \left( \text{dist}(y, L_D) + \text{dist}(x, L_D) \right) d\mu(y) \right)^2
\]

\[
\lesssim |m_D^n|^2 \left( \ell(D)^{-n-1} \int_{C_D} \left( \text{dist}(y, L_D) + \text{dist}(x, L_D) \right) d\mu(y) \right)^2
\]

\[
\lesssim |m_D^n|^2 \left( \beta_{1, \mu}(D)^2 + \alpha_\mu(D)^2 + \left( \frac{\text{dist}(x, L_D)}{\ell(D)} \right)^2 \right). \tag{5.3.30}
\]

Let us consider \( U_{2m}(x) \) now. We can assume that \( \nu_x \) is absolutely continuous with respect to \( \mathcal{H}^n_{L_D} \) (for example, by convolving it with an approximation of the identity and making a limiting argument). Let \( h_x \) be the corresponding density, so \( \nu_x = h_x \mathcal{H}^n_{L_D} \). We may assume that \( h_x \in L^2(\mathcal{H}^n_{L_D}) \). Then,

\[
U_{2m}(x) = (m_D^n f) \int_{A_m(x)} K(x - y) d\nu_x(y) = (m_D^n f) \int_{A_m(x)} K(x - y) h_x (y) d\mathcal{H}^n_{L_D}(y).
\]

Let \( \mathcal{D}^n_{x,0} \) be a fixed dyadic lattice of the \( n \)-plane \( L^2_{D_x} \), and let \( \{\psi_Q\}_{Q \in \mathcal{D}^n_{x,0}} \) be a wavelet basis as the one introduced in Definition 4.1.4 of Chapter 3, but defined on \( L_{D_x} \). Denote by \( E_{D_x} \) the \( n \)-dimensional vector space which defines \( L^2_{D_x} \), and let \( \{Q_k\}_{k \in \mathbb{Z}} \) be a fixed sequence of nested dyadic cubes in \( E_{D_x} \) having the origin as a common vertex and such that \( \ell(Q_k) = 2^{-k} \) for all \( k \in \mathbb{Z} \). Given \( s \in \mathbb{E}_{D_x}^n \), set \( \mathcal{D}^n_{s,0} := \{s + Q : Q \in \mathcal{D}^n_{x,0}\} \) (notice that, for any \( k \in \mathbb{Z} \), the family \( \{Q \in \mathcal{D}^n_{s,0} : \ell(Q) = 2^{-k}\} \) is periodic in the parameter \( s \)). For any \( Q \in \mathcal{D}^n_{x,0} \) and \( y \in L_{D_x} \), if \( y' = s + Q \in \mathcal{D}^n_{x,0} \), we define \( \psi_{Q'}(y) \equiv \psi_{s+Q}(y) := \psi_{Q}(y - s) \). Then \( \{\psi_{Q'}\}_{Q' \in \mathcal{D}^n_{s,0}} \) is also a wavelet basis defined on \( L^2_{D_x} \). Consider the decomposition of \( h_x \) with respect to this basis,

\[
h_x = \sum_{Q \in \mathcal{D}^n_{s,0}} \Delta^\psi_{Q} h_x = \sum_{Q \in \mathcal{D}^n_{s,0}} \Delta^\psi_{Q,s} h_x, \tag{5.3.31}
\]

where \( \Delta^\psi_{Q,s} h_x(z) := \left( \int h_x(y) \psi_{Q}(y - s) d\mu(y) \right) \psi_{Q}(z - s) \) (recall that, for any \( Q \in \mathcal{D}^n_{s,0} \), \( \int \psi_{Q} d\mathcal{H}^n_{L_{D_x}} = 0 \)). We set \( J(Q_S) := -\log_2(\ell(Q_S)) \), and given \( Q \in \mathcal{D}^n_{s,0} \), we set \( J(Q) := -\log_2(\ell(Q)) \) and \( J'(Q) := \max\{J(Q_S), J(Q)\} \). Given \( \Omega \subset \mathbb{E}_{D_x}^n \), denote by \( m_{s \in \Omega} g \) the average of a function \( g : E_{D_x}^n \to \mathbb{R} \) over all \( s \in \Omega \) and with respect to \( \mathcal{H}^n_{L_{D_x}} \). Then, by the periodicity of \( \{\psi_Q\}_{Q \in \mathcal{D}^n_{s,0}} \) in the parameter \( s \) (recall Definition 4.1.4(b)) and (5.3.31), we can write

\[
h_x = \sum_{Q \in \mathcal{D}^n_{s,0}} m_{s \in Q_{J'(Q)}}(h_x) = \sum_{Q \in \mathcal{D}^n_{s,0}} m_{s \in Q_{J'(Q)}}(\Delta^\psi_{Q,s} h_x).
\]

Set \( J := \{Q \in \mathcal{D}^n_{s,0} : \text{supp} \psi_{Q}(\cdot - s) \cap \text{supp} \chi^{2^{j-1}}_{j-1} (x - \cdot) \neq \emptyset \text{ for some } s \in Q_{J'(Q)}\} \). Recall that \( D \in \mathcal{D}_s \) and \( m \in \mathcal{S}_D(x) \). Since \( x \in D \) and \( \ell(D) = 2^{-j} \), if \( Q \in J \), then \( D \subset B(x, C_\alpha \ell(Q)) \) or \( Q \subset B(x, C_\alpha \ell(D)) \) for some constant \( C_\alpha > 0 \) big enough. In particular, if \( \ell(Q) \gtrsim \ell(D) \) then
$D \subset B(z_Q, C_a \ell(Q))$, and if $\ell(Q) \leq C \ell(D)$ with $C > 0$ small enough then $Q \subset B(z_D, C_a \ell(D))$ and $\text{dist}(x, \text{supp} \psi_Q(x - s)) \gtrsim \ell(D)$ for all $s \in Q^0_{J(Q)}$, where $z_Q$ denotes the center of $Q \subset L^2_D$ and $z_D$ denotes the center of $D \subset D$.

We define $J_1 := \{Q \in J : \ell(Q) \leq C \ell(D)\} \subset \{Q \in D^0_x : Q \subset B(z_D, C_a \ell(D))\}$ and $J_2 := J \setminus J_1 \subset \{Q \in D^0_x : D \subset B(z_Q, C_a \ell(Q))\}$, thus dist$(x, \text{supp} \psi_Q(x - s)) \gtrsim \ell(D)$ for all $Q \in J_1$ and $s \in Q^0_{J(Q)}$. Then, using the previous comments, that sup$p\chi_{2^{-j-1}}^m(x - \cdot) \subset \text{supp}\chi_{2^{-j}}^m(x)$ for all $m \in S_D(x)$, that $\int_{A_m(x)} K(x - y) \, d\mathcal{H}^n_{L^2_D}(y) = 0$ by antisymmetry, and that $J'(Q) = J(Q)$ for all $Q \in J_1$ (because $D \subset Q_s$), if $x'$ denotes some fixed point in $A(x, 2^{-j-1}, 2^{-j}) \cap L^2_D$, we have

$$U2_m(x) = (m^2_D f) \sum_{Q_j \in J} K(x, y) \sum_{Q_j \in J} m_{s(Q)}(\Delta^2_{Q, s}(h(x)) \, d\mathcal{H}^n_{L^2_D}(y))$$

$$= \sum_{Q \in J_1} \left| \sum_{Q_j \in J} m_{s(Q)}(\Delta^2_{Q, s}(h(x)) \, d\mathcal{H}^n_{L^2_D}(y)) \right|$$

$$=: U3_m(x) + U4_m(x).$$

(5.3.32)

Let us estimate $U4_m(x)$. By property (e) of the wavelet basis in Definition 4.1.4, we have $|\Delta^2_{Q, s}(h(x)) - \Delta^2_{Q, s}(h(x'))| \leq \|\nabla(\Delta^2_{Q, s}(h(x))\|_{\infty}|x' - y| \lesssim \|\Delta^2_{Q, s}(h(x))\|_{2}|x' - y\| \ell(D)^{-n/2-1}$. Moreover, if $y \in A_m(x)$, then $|x' - y| \lesssim \ell(D)$. Therefore,

$$U4_m(x) \lesssim \sum_{Q \in J_2} \left| \sum_{Q_j \in J} m_{s(Q)}(\Delta^2_{Q, s}(h(x)) \, d\mathcal{H}^n_{L^2_D}(y)) \right|$$

$$\lesssim \sum_{Q \in J_2} \left| \sum_{Q_j \in J} m_{s(Q)}(\Delta^2_{Q, s}(h(x)) \, d\mathcal{H}^n_{L^2_D}(y)) \right|$$

and then, by Cauchy-Schwarz inequality and since $J_2 \subset \{Q \in D^0_x : D \subset B(z_Q, C_a \ell(Q))\}$,

$$\lesssim \left( \sum_{Q_j \in J_2} m_{s(Q)}(\Delta^2_{Q, s}(h(x)) \, d\mathcal{H}^n_{L^2_D}(y)) \right)^2$$

$$\leq \left( \sum_{Q_j \in J_2} m_{s(Q)}(\Delta^2_{Q, s}(h(x)) \, d\mathcal{H}^n_{L^2_D}(y)) \right)^2$$

$$\lesssim \left( \sum_{Q_j \in J_2} \left( \frac{\ell(D)}{\ell(Q)} \right)^2 \right) \left( \sum_{Q_j \in J_2} m_{s(Q)}(\Delta^2_{Q, s}(h(x)) \, d\mathcal{H}^n_{L^2_D}(y)) \right)^2$$

$$\lesssim \left( \sum_{Q_j \in J_2} \left( \frac{\ell(D)}{\ell(Q)} \right)^2 \right) \left( \sum_{Q_j \in J_2} m_{s(Q)}(\Delta^2_{Q, s}(h(x)) \, d\mathcal{H}^n_{L^2_D}(y)) \right)^2$$

(5.3.33)
We are going to estimate \( U^{3m}_{p}(x) \) with techniques very similar to the ones used in Subsections 5.3.4.1 and 5.3.4.3. First of all, let \( b_{*} > 0 \) be a small constant which will be fixed later on, and consider the family \( \mathcal{P} := \{ Q \in \mathcal{D}^{p}_{n,0} : \ell(Q) \leq \ell(D) \} \). Let \( \text{Stp} \) denote the set of cubes \( Q \in \mathcal{P} \) such that there exists \( R_{Q} \in \mathcal{D} \) with \( \ell(R_{Q}) = \ell(Q) \) and such that \( 10R_{Q} \cap (p^{x})^{-1}(\text{supp} \psi_{Q}) \neq \emptyset \), and

\[
\sum_{R \in \mathcal{D} : R \subseteq R_{Q}, \ell(R) \leq \ell(D)} \alpha_{\mu}(10R) \geq b_{*} \quad \text{but} \quad \sum_{R \in \mathcal{D} : P(R) \subseteq R_{Q}, \ell(R) \leq \ell(D)} \alpha_{\mu}(10R) < b_{*}. \tag{5.3.34}
\]

Observe that if \( Q \) and \( Q' \) are different and belong to \( \text{Stp} \), then \( Q \cap Q' = \emptyset \). Notice also that \( D \not\subseteq \text{Stp} \) because we assumed \( \alpha_{\mu}(10D) < C_{*} \). Finally, denote by \( \text{Tr} \) the set of cubes \( Q \in \mathcal{P} \setminus \text{Stp} \) such that \( R \not\subseteq \text{Stp} \) for all \( R \in \mathcal{P} \) with \( R \supset Q \). Then \( \mathcal{P} = \text{Tr} \cup \bigcup_{Q \in \text{Stp}} \{ R \in \mathcal{P} : R \subset Q \} \). By taking \( C_{*} \) small enough we can assume that, if \( R \in J_{1} \cap \mathcal{P} \) and \( R \subset Q \) for some \( Q \in \text{Stp} \), then \( Q \in J_{1} \). So we write

\[
\sum_{Q_{s} \in \mathcal{J}_{Q}(\mathcal{P})} m_{s} \in C_{Q}(\Delta_{Q_{s}, h_{x}}) = \sum_{Q_{s} \in \mathcal{J}_{Q}(\mathcal{P})} m_{s} \in C_{Q}(\Delta_{Q_{s}, h_{x}}) + \sum_{Q_{s} \in \mathcal{J}_{Q}(\mathcal{P})} \sum_{R \in \mathcal{J}_{Q}(\mathcal{P})} m_{s} \in C_{Q}(\Delta_{R_{s}, h_{x}})
\]

Set \( \tilde{\Delta}_{Q_{s}, h_{x}} := \sum_{R \in \mathcal{P} : R \subset Q} \Delta_{R_{s}, h_{x}} \). Then, using the definition of \( J_{1} \) and \( J \), we can split

\[
U^{3m}_{p}(x) = (m^{a}_{D} f) \int_{A_{m}(x)} K(x - y) \sum_{Q_{s} \in \mathcal{J}_{Q}} m_{s} \in C_{Q}(\Delta_{Q_{s}, h_{x}}(y)) d\mathcal{H}^{n}_{L_{D}}(y) + (m^{a}_{D} f) \int_{A_{m}(x)} K(x - y) \sum_{Q_{s} \in \mathcal{J}_{Q}} m_{s} \in C_{Q}(\Delta_{Q_{s}, h_{x}}(y)) d\mathcal{H}^{n}_{L_{D}}(y) \tag{5.3.35}
\]

\[
=: U^{3m}_{p}(x) + U^{3b}_{p}(x).
\]

Notice that, since \( L_{D}^{2} \) is flat and contains \( x \), we have \( \mathcal{H}^{n}_{L_{D}}(A_{m}(x)) \lesssim (\epsilon_{m} - \epsilon_{m+1}) \ell(D)^{n-1} \). Moreover, \( m_{s} \in C_{Q}(\Delta_{Q_{s}, h_{x}}) \) is a function supported in \( CQ \) and has vanishing integral, because the same holds for each \( \Delta_{Q_{s}, h_{x}} \) with \( s \in Q_{Q}(\mathcal{P}) \). Hence, the sum \( \sum_{m \in S_{D}(x)} |U^{3a}_{p}(x)|^{2} \) can be estimated using arguments very similar to the ones in Subsection 4.1.2.2 (see (4.1.16)) or in Subsection 5.3.4.1, but now Lemma 4.1.9 or Claims 5.3.3 and 5.3.4 are not necessary, because we are integrating with respect to \( \mathcal{H}^{n}_{L_{D}} \). One obtains the expected estimate

\[
\sum_{m \in S_{D}(x)} |U^{3a}_{p}(x)|^{2} \lesssim |m^{a}_{D} f|^{2} \sum_{Q \in \mathcal{J}_{Q}} \left( \frac{\ell(Q)}{\ell(D)} \right)^{1/2} \| m_{s} \in C_{Q}(\Delta_{Q_{s}, h_{x}}) \|_{2}^{2} \ell(D)^{-n} \tag{5.3.36}
\]

(for simplicity of notation, we have set \( \| \cdot \|_{p} := \| \cdot \|_{L_{p}(\mathcal{H}^{n}_{L_{D}})} \)).

Since \( m_{s} \in C_{Q}(\Delta_{Q_{s}, h_{x}}) \) has vanishing integral and it is supported in a neighborhood of \( Q \), the term \( U^{3b}_{p}(x) \) can be estimated in the same manner (but without using the estimate
\[ \|m_{s\in Q^0_{j(q)}}(\tilde{\Delta}^\psi_{Q,s}h_x)\|_1^2 \leq \ell(Q)^n \|m_{s\in Q^0_{j(q)}}(\tilde{\Delta}^\psi_{Q,s}h_x)\|_2^2, \] and one obtains
\[
\sum_{m\in \mathcal{D}_D(x)} |U^\psi_{m}(x)|^2 \lesssim |m_Df|^2 \sum_{Q\in J_0\cap\text{Stp}} \left( \frac{\ell(Q)}{\ell(D)} \right)^{1/2} \|m_{s\in Q^0_{j(q)}}(\tilde{\Delta}^\psi_{Q,s}h_x)\|_1^2 \ell(D)^{-n} \ell(Q)^{-n}. \tag{5.3.37}
\]

At this point, one must adapt Lemma 4.1.6 to the general case of a uniformly rectifiable measure instead of the particular case of the Hausdorff measure on a Lipschitz graph with small slope. Recall that we have fixed \( x \in D \subseteq S \setminus \text{Trs} \), and we denote by \( Q_S \) the maximal cube in \( S \) from the corona decomposition, so \( D \subset Q_S \).

**Lemma 5.3.9.** Assume that \( \alpha_\mu(D) < C_* \), for some constant \( C_* > 0 \) small enough. Given \( Q \in \mathcal{D}^{n,0}_x \), there exists constants \( C_1, C_2 > 1 \) depending on \( C_* \) and \( b_* \) such that,

(a) if \( Q \in J_3 \) and \( \ell(Q) > \ell(Q_S) \), then \( m_{s\in Q^0_{j(q)}}(\|\Delta^\psi_{Q,s}h_x\|_2) \lesssim \ell(Q_S)^n \ell(Q)^{-n/2} \),

(b) if \( Q \in J_2 \) and \( \ell(Q) \leq \ell(Q_S) \), then
\[
m_{s\in Q^0_{j(q)}}(\|\Delta^\psi_{Q,s}h_x\|_2) \lesssim \left( \sum_{R \in \mathcal{D} : D \subset R \subset B(z_Q,C_1 \ell(Q))} \alpha_\mu(C_1 R) + \frac{\text{dist}(x,L_D)}{\ell(D)} \right) \ell(Q)^{n/2},
\]

(c) if \( Q \in J_1 \cap \text{Tr} \), then there exists \( Q_0 \equiv Q_0(x,Q) \in \mathcal{D} \) depending on \( x \) and \( Q \in \mathcal{D}^{n,0}_x \) such that \( Q_0 \subset C_2 D \), \( \ell(Q_0) \approx \ell(Q) \), \( Q_0 \cap (p^{-1}(\text{supp } \psi_Q)) \neq \emptyset \) and
\[
m_{s\in Q^0_{j(q)}}(\Delta^\psi_{Q,s}h_x) \lesssim \left( \sum_{R \in \mathcal{D} : D \subset R \subset C_2 D} \alpha_\mu(C_2 R) + \frac{\text{dist}(x,L_D)}{\ell(D)} \right) \ell(Q)^{n/2}, \quad \text{and}
\]

(d) if \( Q \in J_1 \cap \text{Stp} \), then \( m_{s\in Q^0_{j(q)}}(\tilde{\Delta}^\psi_{Q,s}h_x) \|_1 \lesssim \ell(Q)^n \).

**Proof of Lemma 5.3.9(a).** By Definition 4.1.4(e), for any \( s \in Q^0_{j(q)} \) we have
\[
\|\Delta^\psi_{Q,s}h_x\|_\infty \lesssim \langle h_x, \psi_{s+Q} \rangle |\ell(Q)|^{-n/2} \lesssim (\ell(Q))^{-n} \int h_x dH^0_{L_D} = (\ell(Q))^{-n} \int d\nu_x
\]
\[
= (\ell(Q))^{-n} \int d(\mu^x(x40Q_s \mu)) = (\ell(Q))^{-n} \int_{40Q_s} d\mu \lesssim \frac{\ell(Q)^n}{\ell(Q)}. \]

Hence, \( \|\Delta^\psi_{Q,s}h_x\|_2 \leq \|\Delta^\psi_{Q,s}h_x\|_\infty \ell^n \text{L}^{n/2}(\text{supp } \psi_{s+Q})^{1/2} \lesssim \ell(Q)^n \ell(Q)^{-n/2} \) for all \( s \in Q^0_{j(q)} \), and Lemma 5.3.9(a) follows by taking the average over \( s \in Q^0_{j(q)}. \)

**Proof of Lemma 5.3.9(b).** Since \( D \subset B(z_Q,C_a \ell(Q)) \), \( D \subseteq S \), and \( \ell(D) \lesssim \ell(Q) \lesssim \ell(Q_S) \), by taking \( C_{\text{cor}} \) big enough (see property (f) in Subsection 5.3.2), we can assume that \( \mu \) is well approximated by \( \Gamma_S \) in all the intermediate cubes between \( D \) and \( B(z_Q,C_a \ell(Q)) \). In this case, taking into account the comments above (5.3.29), similar arguments to the ones used in the proof of Lemma 4.1.6(a) applies for each \( s \in Q^0_{j(q)} \), and Lemma 5.3.9(b) follows by taking the average over \( s \in Q^0_{j(q)}. \)
Proof of Lemma 5.3.9(c). Given $Q \in J_1 \cap \text{Tr}$, using (5.3.34) we have

$$\sum_{R' \in \mathcal{D}: R \subset R', \ell(R') \leq \ell(D)} \alpha_{\mu}(10R') < b_s$$

for all $R \in \mathcal{D}$ with $\ell(R) = \ell(Q)$ and such that $R \cap (p^x)^{-1}(\text{supp } \psi_{s+Q}) \neq \emptyset$ for all $s \in Q_j(Q)$. Taking $b_s$ small enough, we can use similar arguments to the ones in the proof of Lemma 4.1.6(b) (see (4.1.29) and below), and we obtain

$$\|\Delta^\psi_{Q,s} h_x\|_2 \lesssim \left( \sum_{R \in \mathcal{D}: R \subset R', \ell(R') \leq \ell(D)} \alpha_{\mu}(C_2R) + \frac{\text{dist}(x, L_D)}{\ell(D)} \right) \ell(Q)^{n/2}$$

for some $Q_0(x, Q) \in \mathcal{D}$ and all $s \in Q_j(Q)$. As before, Lemma 5.3.9(c) follows by taking the average over $s \in Q_j(Q)$, and noting that $\|m_{s \in Q_j(Q)}(\Delta^\psi_{Q,s} h_x)\|_2 \leq m_{s \in Q_j(Q)}\|\Delta^\psi_{Q,s} h_x\|_2$ by Minkowski’s integral inequality.

Proof of Lemma 5.3.9(d). This is the key point where taking averages of dyadic lattices (with respect to the parameter $s$) is necessary. More precisely, in the proof of Lemma 5.3.9(a), . . . , (c) we obtained estimates independent of $s$ and then we made an average. However, in Lemma 5.3.9(d) the estimate that we want to prove might not hold for a particular choice of $s$ but, as we will see, it holds in average. Given $Q \in J_1 \cap \text{Stp}$, we have to show that $\|m_{s \in Q_j(Q)}(\Delta^\psi_{Q,s} h_x)\|_1 \lesssim \ell(Q)^n$. Fix $s \in Q_j(Q)$. Recall that

$$\tilde{\Delta}^\psi_{Q,s} h_x = \sum_{R \in \mathcal{P}: R \subset Q} \Delta^\psi_{R,s} h_x$$

where we have set $I_s := I_{s+Q}^1 + I_{s+Q}^2$. We are going to estimate $I_s$, $I_{s+Q}$, and $\tilde{\Delta}^\psi_{Q,s} h_x$ separately. For the case of $I_s$, we have

$$\chi_{s+Q} h_x = \sum_{R \in \mathcal{D}^{0}_{s+Q}: \ell(R) > \ell(Q)} \Delta^\psi_{R,s} h_x + \sum_{R \in \mathcal{D}^{0}_{s+Q}: \ell(R) \leq \ell(Q)} \Delta^\psi_{R,s} h_x = \chi_{s+Q} I'_s + I_s,$$

where we have set $I'_s := \sum_{R \in \mathcal{D}^{0}_{s+Q}: \ell(R) > \ell(Q)} \Delta^\psi_{R,s} h_x$. On one hand, since $Q \in J_1 \cap \text{Stp}$, (5.3.34) holds. Thus, using that $\sum_{R \in \mathcal{D}} (\sum_{R \in \mathcal{D}} \alpha_{\mu}(10R') < b_s$, one can show that

$$\|\chi_{s+Q} h_x\|_1 \lesssim \ell(Q)^n$$

(see (4.1.29) and below for a related argument). On the other hand, since $\|\chi_{s+Q} h_x\|_1 \lesssim \ell(Q)^n$, it is known that then $\|\chi_{s+Q} I'_s\|_1 \lesssim \ell(Q)^n$ (see [Da1, Part I], in particular pay attention to the last sum in equation (46) of Part I). Combining these estimates, we conclude that $\|I_s\|_1 \lesssim \ell(Q)^n$. 
Let us now deal with $II_s$. First of all, split $II_s$ into different scales, that is

$$
\sum_{R \in \mathcal{P}: \supp \psi_R \cap Q \neq \emptyset} \chi_{s+Q} \Delta^\psi_{R,s} h_x = \sum_{k \geq J(Q)} \sum_{R \in \mathcal{P}: \ell(R) = 2^{-k}} \chi_{s+Q} \Delta^\psi_{R,s} h_x.
$$

Observe that if $k \geq J(Q)$, $\supp \psi_R \cap Q \neq \emptyset$, $\ell(R) = 2^{-k}$, and $R \not\subset Q$, then $s + R \subset U_{C2^{-k}}(s + \partial Q)$, where $C > 1$ is some fixed constant and $U_{C2^{-k}}(s + \partial Q) := \{ z \in L^2_D : \text{dist}(z, s + \partial Q) < C2^{-k} \}$. Hence, using Definition 4.1.4(e) and the definition of $h_x$, we have

$$
\| II_s \|_1 \leq \sum_{k \geq J(Q)} \sum_{R \in \mathcal{P}: \supp \psi_R \cap Q \neq \emptyset} \| \Delta^\psi_{R,s} h_x \|_1 \lesssim \sum_{k \geq J(Q)} \nu_x(U_{C2^{-k}}(s + \partial Q)).
$$

The case of $III_s$ can be dealt with very similar techniques, and one obtains the same estimate. Therefore,

$$
\| m_{s \in Q^+_j(Q)}(\Delta^\psi_{Q,s} h_x) \|_1 = \| m_{s \in Q^+_j(Q)}(I_s + II_s + III_s) \|_1 = m_{s \in Q^+_j(Q)}|I_s + II_s + III_s|_1
\lesssim \ell(Q)^n + m_{s \in Q^+_j(Q)} \left( \sum_{k \geq J(Q)} \nu_x(U_{C2^{-k}}(s + \partial Q)) \right). \tag{5.3.39}
$$

Using Fubini’s theorem, it is not difficult to show that

$$
m_{s \in Q^+_j(Q)} \nu_x(U_{C2^{-k}}(s + \partial Q)) \lesssim 2^{-k} \ell(Q)^{-1} \nu_x(CQ)
$$

for all for $k \geq J(Q)$ (see [To8, Lemma 7.5] for example, for a related argument). Since $Q \in \text{Sth}$, then (5.3.34) holds and then, as in (5.3.38), we have $\nu_x(CQ) \lesssim \ell(Q)^n$ (by choosing the constants suitably), thus

$$
m_{s \in Q^+_j(Q)} \left( \sum_{k \geq J(Q)} \nu_x(U_{C2^{-k}}(s + \partial Q)) \right) \lesssim \ell(Q)^n.
$$

If we combine this last estimate with (5.3.39), we are done. \hfill \Box

We are ready to put all the estimates together to bound the first term on the right hand side of (5.3.24). From (5.3.27), (5.3.29), (5.3.32), and (5.3.35) we have

$$
\sum_{m \in \mathcal{S}_D(x)} \left| \langle K \chi_{\ell(x+1)} \ast (m_D^n \chi_{\beta \mu}) \rangle(x) \right|^2 \lesssim |m_D^n f|^2 \alpha_\mu(10D)^2 + \sum_{m \in \mathcal{S}_D(x)} \left( |U_{1_m}(x)|^2 + |U_{3_m^a}(x)|^2 + |U_{3_m^b}(x)|^2 + |U_{4_m}(x)|^2 \right), \tag{5.3.40}
$$

Let us deal with $U_{1_m}(x)$ (the term $|m_D^n f|^2 \alpha_\mu(10D)^2$ above is handled in the same manner). If $L_D^1$ and $L_D^2$ denote a minimizing $n$-plane for $\beta_{1,\mu}(D)$ and $\beta_{2,\mu}(D)$, respectively, one can show
that \( \text{dist}_H(L_D \cap B_D, L_D^1 \cap B_D) \lesssim \alpha_\mu(D) \ell(D) \) and \( \text{dist}_H(L_D^1 \cap B_D, L_D^2 \cap B_D) \lesssim \beta_\mu(D) \ell(D) \), so we have \( \text{dist}(x, L_D) \lesssim \text{dist}(x, L_D^1) + \beta_\mu(D) \ell(D) + \alpha_\mu(D) \ell(D) \) for \( x \in D \). Then, by (5.3.30) and Carleson’s embedding theorem,

\[
\sum_{S \in \text{Trs}} \sum_{D \in S} \int_D \sum_{m \in S_D(x)} |U1_m|^2 \, d\mu \lesssim \sum_{D \in D} \int_D |m_D \ell f|^2 \left( \beta_{1,\mu}(D)^2 + \alpha_\mu(D)^2 + \left( \frac{\text{dist}(x, L_D)}{\ell(D)} \right)^2 \right) \, d\mu(x)
\]

\[
\lesssim \sum_{D \in D} |m_D \ell f|^2 \ell(D)^n \left( \beta_{1,\mu}(D)^2 + \alpha_\mu(D)^2 + \beta_\mu(D)^2 \right) \lesssim \|f\|_{L^2(\mu)}^2.
\]

For the case of \( U3_m^2(x) \), by (5.3.36) and Lemma 5.3.9(c) applied to the cubes in \( J_1 \cap \text{Tr} \), we have

\[
\sum_{S \in \text{Trs}} \sum_{D \in S} \int_D \sum_{m \in S_D(x)} |U3_m^2|^2 \, d\mu \lesssim \sum_{D \in D} |m_D \ell f|^2 \int_D \sum_{Q \in J_1 \cap \text{Tr}} \left( \frac{\ell(Q)}{\ell(D)} \right)^{n+1/2} \left( \sum_{R \in D: Q \subset R \subset C_2D} \alpha_\mu(C_2R) \right)^2 \, d\mu(x)
\]

\[
+ \sum_{D \in D} |m_D \ell f|^2 \int_D \sum_{Q \in J_1 \cap \text{Tr}} \left( \frac{\ell(Q)}{\ell(D)} \right)^{n+1/2} \left( \frac{\text{dist}(x, L_D)}{\ell(D)} \right)^2 \, d\mu(x) =: S_1 + S_2.
\]

Recall that \( J_1 \subset \{ Q \in \mathcal{D}_{x,s}^n : Q \subset B(z_D, C_\ell(D)) \} \). Then \( \sum_{Q \in J_1} \left( \frac{\ell(Q)}{\ell(D)} \right)^{n+1/2} \lesssim 1 \), and since \( \text{dist}(x, L_D) \lesssim \text{dist}(x, L_D^1) + \beta_{\mu}(D) \ell(D) + \alpha_\mu(D) \ell(D) \) for \( x \in D \), then \( S_2 \lesssim \sum_{D \in D} |m_D \ell f|^2 (\beta_\mu(D)^2 + \alpha_\mu(D)^2) \ell(D)^n \), and hence \( S_2 \leq C \|f\|_{L^2(\mu)}^2 \), by Carleson’s embedding theorem. For \( S_1 \), since \( \ell(Q) \approx \ell(Q_0(x, Q)) \) (recall the definition of \( Q_0 \equiv Q_0(x, Q) \) in Lemma 5.3.9(c)), \( Q_0(x, Q) \subset C_2D \), and every \( Q_0 \in \mathcal{D} \) intersects \((p^s)^{-1}(Q)\) for finitely many cubes \( Q \in \mathcal{D}_{x,0}^n \) (with a bound for the number of such cubes \( Q \) independent of \( x \) and \( Q_0 \)), we have

\[
\sum_{Q \in J_1 \cap \text{Tr}} \left( \frac{\ell(Q)}{\ell(D)} \right)^{n+1/2} \left( \sum_{R \in D: Q \subset R \subset C_2D} \alpha_\mu(C_2R) \right)^2
\]

\[
= \sum_{P \in \mathcal{D}: P \subset C_2D} \sum_{Q \in \mathcal{D}_{x,0}^n: Q \subset B(z_D, C_\ell(D)), Q_0(x, Q) = P} \left( \frac{\ell(Q)}{\ell(D)} \right)^{n+1/2} \left( \sum_{R \in D: P \subset R \subset C_2D} \alpha_\mu(C_2R) \right)^2
\]

\[
\lesssim \sum_{P \in \mathcal{D}: P \subset C_2D} \left( \frac{\ell(P)}{\ell(D)} \right)^{n+1/2} \left( \sum_{R \in D: P \subset R \subset C_2D} \alpha_\mu(C_2R) \right)^2.
\]
By Cauchy-Schwarz inequality,

\[
\sum_{P \in \mathcal{D} : P \subset C_2 D} \left( \frac{\ell(P)}{\ell(D)} \right)^{n+1/2} \left( \sum_{R \in \mathcal{D} : P \subset R \subset C_2 D} \alpha_\mu(C_2 R) \right)^2 \lesssim \sum_{P \in \mathcal{D} : P \subset C_2 D} \left( \frac{\ell(P)}{\ell(D)} \right)^{n+1/2} \log_2 \left( \frac{\ell(D)}{\ell(P)} \right) \sum_{R \in \mathcal{D} : P \subset R \subset C_2 D} \alpha_\mu(C_2 R)^2
\]

\[
\lesssim \sum_{P \in \mathcal{D} : P \subset C_2 D} \left( \frac{\ell(P)}{\ell(D)} \right)^{n+1/4} \sum_{R \in \mathcal{D} : P \subset R \subset C_2 D} \alpha_\mu(C_2 R)^2 \quad \text{(5.3.41)}
\]

\[
\lesssim \sum_{R \in \mathcal{D} : R \subset C_2 D} \alpha_\mu(C_2 R)^2 \sum_{P \in \mathcal{D} : P \subset R} \left( \frac{\ell(P)}{\ell(D)} \right)^{n+1/4}
\]

\[
\lesssim \sum_{R \in \mathcal{D} : R \subset C_2 D} \alpha_\mu(C_2 R)^2 \left( \frac{\ell(R)}{\ell(D)} \right)^{n+1/4} =: \lambda_1(D)^2.
\]

In (4.1.36) we showed that this \( \lambda_1 \) coefficients satisfy a Carleson packing condition, so by (5.3.41) and Carleson’s embedding theorem we obtain \( S_1 \lesssim \sum_{D \in \mathcal{D}} |m^\mu_D f|^2 \ell(D)^n \lambda_1(D)^2 \lesssim \|f\|_{L^2(\mu)}^2 \), which, combined with \( S_2 \lesssim \|f\|_{L^2(\mu)}^2 \), yields \( \sum_{S \in \mathcal{S}} \sum_{D \in \mathcal{D}} \int_D \sum_{m \in \mathcal{S}_D(x)} |U^3_m|^2 \, d\mu \lesssim \|f\|_{L^2(\mu)}^2 \).

Let us deal now with \( U^3_m \). By (5.3.37) and Lemma 5.3.9(d) applied to the cubes in \( J_1 \cap \text{Stp} \), we have

\[
\sum_{S \in \mathcal{S}} \sum_{D \in \mathcal{D}} \int_D \sum_{m \in \mathcal{S}_D(x)} |U^3_m|^2 \, d\mu \lesssim \sum_{D \in \mathcal{D}} |m^\mu_D f|^2 \int_D \sum_{Q \in \mathcal{J}_1 \cap \text{Stp}} \left( \frac{\ell(Q)}{\ell(D)} \right)^{1/2} \|m_{s \in \mathcal{Q}_{j(x)}} (\tilde{\Delta}^\psi_{Q_s} h_x)\|_1 \, d\mu(x)
\]

\[
\lesssim \sum_{D \in \mathcal{D}} |m^\mu_D f|^2 \int_D \sum_{Q \in \mathcal{J}_1 \cap \text{Stp}} \left( \frac{\ell(Q)}{\ell(D)} \right)^{n+1/2} \, d\mu.
\]

Given \( D \in \mathcal{D} \), consider the family \( \Lambda_D := \{ R \in \mathcal{D} : R = R_Q \text{ for some } x \in D \text{ and some } Q \in J_1 \cap \text{Stp} \} \) (see the definition of \( R_Q \) in (5.3.34)). Observe that every \( R \in \mathcal{D} \) intersects \( (p^*)^{-1}(Q \cap L^p_D) \) for finitely many cubes \( Q \in \mathcal{D}_x^{n,0} \) such that \( \ell(Q) = \ell(R) \). Thus, similarly to what we did for \( Q \in J_1 \cap \text{Tr} \) in the case of \( U^3_m \), we have

\[
\sum_{D \in \mathcal{D}} |m^\mu_D f|^2 \int_D \sum_{Q \in \mathcal{J}_1 \cap \text{Stp}} \left( \frac{\ell(Q)}{\ell(D)} \right)^{n+1/2} \, d\mu \lesssim \sum_{D \in \mathcal{D}} |m^\mu_D f|^2 \int_D \sum_{R \in \Lambda_D} \left( \frac{\ell(R)}{\ell(D)} \right)^{n+1/2} \, d\mu
\]

\[
\lesssim \sum_{D \in \mathcal{D}} |m^\mu_D f|^2 \sum_{R \in \Lambda_D} \left( \frac{\ell(R)}{\ell(D)} \right)^{n+1/2} \mu(D) = \sum_{D \in \mathcal{D}} |m^\mu_D f|^2 \lambda_2(D)^2 \mu(D),
\]

where we have set \( \lambda_2(D)^2 := \sum_{R \in \Lambda_D} (\ell(R)/\ell(D))^{n+1/2} \). Since the \( \alpha_\mu \)'s satisfy a Carleson packing condition, it is not hard to show that the same holds for the \( \lambda_2 \)'s. More precisely,
since for any \( R \in \Lambda_D \) we have \( \sum_{R' \in \Delta: R \subset R', \ell(R) \leq \ell(D)} \alpha_{\mu}(10R') \geq b_\ast \) by (5.3.34), then

\[
\lambda_2(D)^2 \leq b_\ast^2 \sum_{R \in \Lambda_D} \left( \frac{\ell(R)}{\ell(D)} \right)^{n+1/2} \left( \sum_{R' \in \Delta: R \subset R', \ell(R) \leq \ell(D)} \alpha_{\mu}(10R') \right)^2,
\]

and we can proceed as in (5.3.41). Hence, putting these estimates together and using Carleson’s embedding theorem on the \( \lambda_2 \)'s, we obtain \( \sum_{S \in \text{Trs}} \sum_{D \in S} |U_{3m}b|^2 \, d\mu \lesssim \|f\|^2_{L^2(\mu)} \).

We deal now with \( U_{4m}(x) \). By (5.3.33) and Lemma 5.3.9(a) and (b) applied to the cubes in \( J_2 \),

\[
\sum_{S \in \text{Trs}} \sum_{D \in S} \int_{D} \sum_{m \in S_D(x)} |U_{4m}|^2 \, d\mu \lesssim \sum_{S \in \text{Trs}} \sum_{D \in S} |m_D^{\mu} f|^2 \int_{D} \sum_{Q \in J_2} \left( \frac{\ell(D)}{\ell(Q)} \right)^{1/2} m_{S \ell(Q)} \left( \|\Delta_{Q}^{\psi} h_2\|^2 \right) \frac{d\mu}{\ell(Q)^n} \leq \sum_{S \in \text{Trs}} \sum_{D \in S} |m_D^{\mu} f|^2 \int_{D} \sum_{Q \in J_2: \ell(Q) > \ell(Q_S)} \left( \frac{\ell(D)}{\ell(Q)} \right)^{1/2} \left( \frac{\ell(Q_S)}{\ell(Q)} \right)^{2n} d\mu =: S_3 + S_4.
\]

For the case of \( S_3 \), since \( \text{dist}(x, L_D) \lesssim \text{dist}(x, L_D^2) + \beta_{2,\mu}(D) \ell(D) + \alpha_{\mu}(D) \ell(D) \) for \( x \in D \) and \( \sum_{Q \in J_2} (\ell(D)/\ell(Q))^{1/2} \lesssim 1 \), the second term in the definition of \( \alpha_{\mu}(C_1R) \) is bounded by \( \sum_{D \in D} |m_D^{\mu} f|^2 (\beta_{2,\mu}(D)^2 + \alpha_{\mu}(D)^2) (\ell(D))^{n} \), and hence by \( C \|f\|^2_{L^2(\mu)} \), by Carleson’s embedding theorem. For the first term defining \( \lambda_2 \), by Cauchy-Schwarz inequality,

\[
\sum_{S \in \text{Trs}} \sum_{D \in S} |m_D^{\mu} f|^2 \int_{D} \sum_{Q \in J_2: \ell(Q) \leq \ell(Q_S)} \left( \frac{\ell(D)}{\ell(Q)} \right)^{1/2} \left( \sum_{R \in \Delta^D: D \subset R \subset \ell(Q), \ell(R) \leq \ell(Q)} \alpha_{\mu}(C_1R)^2 \right) d\mu \\
\lesssim \sum_{S \in \text{Trs}} \sum_{D \in S} |m_D^{\mu} f|^2 \int_{D} \sum_{Q \in J_2: \ell(Q) \leq \ell(Q_S)} \left( \frac{\ell(D)}{\ell(Q)} \right)^{1/2} \left( \frac{\ell(Q)}{\ell(D)} \right) \log_2 \left( \frac{\ell(Q)}{\ell(D)} \right) \sum_{R \in \Delta^D: D \subset R \subset \ell(Q), \ell(R) \leq \ell(Q)} \alpha_{\mu}(C_1R)^2 d\mu \\
\lesssim \sum_{D \in D} |m_D^{\mu} f|^2 \int_{D} \sum_{Q \in D_{0}^{\mu}, D_{C} \subseteq B(z_Q, C_1 \ell(Q))} \left( \frac{\ell(D)}{\ell(Q)} \right)^{1/4} \sum_{R \in \Delta^D: D \subset R \subset \ell(Q), \ell(R) \leq \ell(Q)} \alpha_{\mu}(C_1R)^2 d\mu \\
\lesssim \sum_{D \in D} |m_D^{\mu} f|^2 \int_{D} \sum_{R \in D_{C} \subseteq B(z_Q, C_1 \ell(Q))} \left( \frac{\ell(D)}{\ell(Q)} \right)^{1/4} \sum_{Q \in D_{0}^{\mu}, D_{C} \subseteq B(z_Q, C_1 \ell(Q))} \alpha_{\mu}(C_1R)^2 d\mu.
\]
Notice that \( \sum_{Q \in \mathcal{D}_x: \ell(Q) \leq \ell(T_S)} \left( \frac{\ell(D)}{\ell(Q)} \right)^{1/4} \lesssim \left( \frac{\ell(D)}{\ell(R)} \right)^{1/4} \), thus

\[
\sum \sum |m_D^{f}|^2 \int_{D} \sum \sum \left( \frac{\ell(D)}{\ell(Q)} \right)^{1/2} \frac{\alpha_{\mu}(C_1 R)}{D \in \mathcal{D}, D \subset B(z_{Q}, C_1 \ell(Q))} \alpha_{\mu}(C_1 R)^2 \frac{\ell(D)}{\ell(R)} \right)^{1/4}
\]

\[
\lesssim \sum \sum |m_D^{f}|^2 \ell(D)^n \sum \sum_{D \in \mathcal{D}, D \subset B(z_{Q}, C_1 \ell(Q))} \alpha_{\mu}(C_1 R)^2 \frac{\ell(D)}{\ell(R)} \right)^{1/4} =: \sum_{D \in \mathcal{D}} |m_D^{f}|^2 \ell(D)^n \lambda_3(D)^2.
\]

(5.3.42)

In (4.1.37) we showed that some coefficients very similar to the \( \lambda_3 \)'s satisfy a Carleson packing condition, so by very similar arguments one can show the Carleson packing condition for the \( \lambda_4 \)'s, and then, by Carleson’s embedding theorem, this last term is bounded by \( C \|f\|_{L^2(\mu)}^2 \).

So we obtain \( S_3 \lesssim \|f\|_{L^2(\mu)}^2 \). The estimate of \( S_4 \) is easier. We have

\[
S_4 = \sum \sum |m_D^{f}|^2 \int_{D} \sum \sum \left( \frac{\ell(D)}{\ell(Q)} \right)^{1/2} \frac{\ell(Q)^{2n}}{\ell(Q)^{2n}} \alpha_{\mu}(C_1 R) \frac{\ell(D)}{\ell(Q)} \right)^{1/4}
\]

\[
\lesssim \sum \sum |m_D^{f}|^2 \ell(D)^n \sum \sum \left( \frac{\ell(D)}{\ell(Q)} \right)^{1/2} \frac{\ell(Q)^{2n}}{\ell(Q)^{2n}} d\mu(x).
\]

As before, \( \sum_{Q \in \mathcal{D}_x: \ell(Q) > \ell(Q_S), D \subset B(z_{Q}, C_1 \ell(Q))} \ell(Q)^{-2n-1/2} \lesssim \ell(Q_S)^{-2n-1/2} \), thus

\[
S_4 \lesssim \sum \sum |m_D^{f}|^2 \int_{D} \sum \sum \left( \frac{\ell(D)}{\ell(Q)} \right)^{1/2} \frac{\ell(Q)^{2n}}{\ell(Q)^{2n}} \alpha_{\mu}(C_1 R)^2 \frac{\ell(D)}{\ell(Q)} \right)^{1/4}
\]

\[
\lesssim \sum \sum |m_D^{f}|^2 \ell(D)^n \sum \sum \left( \frac{\ell(D)}{\ell(Q)} \right)^{1/2} \frac{\ell(Q)^{2n}}{\ell(Q)^{2n}} d\mu(x).
\]

Similarly to the case of the \( \lambda_3 \) coefficients, one can show that the \( \lambda_4 \)'s also satisfy a Carleson packing condition, thus \( S_4 \lesssim \|f\|_{L^2(\mu)}^2 \) by Carleson’s embedding theorem. Actually, if one defines \( \tilde{\alpha}_{\mu}(Q) = 1 \) if \( Q = Q_S \) for some \( S \in \text{Trs} \) and \( \tilde{\alpha}_{\mu}(Q) = 0 \) otherwise, using the packing condition for the cubes \( Q_S \) with \( S \in \text{Trs} \) (see property (e) in Subsection 5.3.2), one can easily verify that the \( \tilde{\alpha}_{\mu} \)'s satisfy a Carleson packing condition. Then,

\[
\lambda_4(D)^2 \leq \sum_{S \in \text{Trs}, D \subset Q_S} \left( \frac{\ell(D)}{\ell(Q_S)} \right)^{1/2} \tilde{\alpha}_{\mu}(Q_S) = \sum_{Q \in \mathcal{D}, D \subset Q} \left( \frac{\ell(D)}{\ell(Q)} \right)^{1/2} \tilde{\alpha}_{\mu}(Q)^2,
\]

and we can argue as in the case of the \( \lambda_3 \)'s in (5.3.42).

By the estimates of \( S_3 \) and \( S_4 \), we obtain

\[
\sum \sum \int_{D} \sum_{m \in \mathcal{S}_D(x)} |U_{4m}|^2 d\mu \lesssim \|f\|_{L^2(\mu)}^2.
\]
Finally, plugging all these estimates for \( U_{1m}, U_{3m}, U_{3m}^c, \) and \( U_{4m} \) in (5.3.40), and combining the result with (5.3.24) and (5.3.25), we conclude that

\[
\sum_{S \in \text{Trs}} \sum_{D \in S} \sum_{m \in S_D(x)} \sum_{R \in V(D)} \left( K\chi_{m+1}^\epsilon \ast ((m_R^tf)\chi_R\mu)(x) \right)^2 \, d\mu(x) \lesssim \|f\|_{L^2(\mu)}^2. \tag{5.3.43}
\]

### 5.3.4.5 Final estimates

From (5.3.14), (5.3.19), and (5.3.43), we obtain the following:

\[
\sum_{S \in \text{Trs}} \sum_{D \in S} \sum_{m \in S_D(x)} \sum_{R \in V(D)} \left( K\chi_{m+1}^\epsilon \ast (\Delta_Q f)\mu)(x) \right)^2 \, d\mu(x) \\
+ \sum_{S \in \text{Trs}} \sum_{D \in S} \sum_{m \in S_D(x)} \sum_{Q \in \text{Tr}(R)} \left( K\chi_{m+1}^\epsilon \ast (\tilde{\Delta}_Q f)(x) \right)^2 \, d\mu(x) \\
+ \sum_{S \in \text{Trs}} \sum_{D \in S} \sum_{m \in S_D(x)} \sum_{R \in V(D)} \left( K\chi_{m+1}^\epsilon \ast (m_R^tf)\chi_R\mu)(x) \right)^2 \, d\mu(x) \lesssim \|f\|_{L^2(\mu)}^2. 
\]

Combining this estimate with (5.3.7) and (5.3.8), we deduce

\[
\sum_{S \in \text{Trs}} \sum_{D \in S} \sum_{m \in S_D(x)} \left| (K\chi_{m+1}^\epsilon \ast (f\mu)(x) \right| \, d\mu(x) \lesssim \|f\|_{L^2(\mu)}^2. 
\]

Finally, using (5.3.4) and (5.3.5), we conclude that

\[
\| (\mathcal{V}_\rho^S \circ \mathcal{T}^\mu) f \|_{L^2(\mu)}^2 \lesssim \sum_{D \in \mathcal{D}} \sum_{m \in S_D(x)} \left| (K\chi_{m+1}^\epsilon \ast (f\mu)(x) \right| \, d\mu(x) \lesssim \|f\|_{L^2(\mu)}^2. 
\]

As we said below (5.3.3), this finishes the proof of Theorem 5.3.1.

### 5.4 \( L^2 \) boundedness of \( \mathcal{V}_\rho \circ \mathcal{R}^\mu \) implies that \( \mu \) is a uniformly rectifiable measure

Let \( C_\mu > 0 \) be the AD regularity constant of an AD regular measure \( \mu \), that is \( C_\mu^{-1}r^a \leq \mu(B(x, r)) \leq C_\mu r^a \) for all \( x \in \text{supp} \mu \) and \( 0 \leq r < \text{diam}(\text{supp} \mu) \). For simplicity of notation, we may assume that \( \text{diam}(\text{supp} \mu) = \infty \) (the general case follows with minor modifications in our arguments).

In this section, we set \( K(x) = x|x|^{n-1} \) for \( x \neq 0 \). Recall that, given \( \epsilon > 0 \), a Borel measure \( \mu \), and \( f \in L^1(\mu) \), we have set \( \mathcal{R}^\mu f := \{ R^\mu_x f \}_{\epsilon > 0} \), where

\[
R^\mu_x f(x) = \int_{|x-y|>\epsilon} K(x-y) f(y) \, d\mu(y).
\]
5.4. $L^2$ boundedness of $\mathcal{V}_\rho \circ R^\mu$ implies that $\mu$ is uniformly rectifiable

In order to prove the main theorem of this section, namely Theorem 5.4.8, we need a known result concerning Riesz transforms and rectifiability, but first we have to introduce some notation.

**Definition 5.4.1** (Special truncation of Riesz transform). For $\epsilon > 0$, let $\varphi_\epsilon$ be as in Definition 5.0.3. For $m \in \mathbb{Z}$ and $x \in \mathbb{R}^d$ denote $\tilde{\varphi}_m(x) = \varphi_{2^{-m-1}}(x) - \varphi_{2^{-m}}(x)$. Given a Borel measure $\mu$, we set

$$S_m \mu(x) = \int \tilde{\varphi}_m(x - y) K(x - y) \, d\mu(y).$$

**Lemma 5.4.2** (Lemma 5.8 of [DS1]). Given $Q \in \mathcal{D}$, there exist $n + 1$ points $x_0, \ldots, x_n$ in $Q$ such that $\text{dist}(x_j, L_{j-1}) \geq C \ell(Q)$, where $L_k$ denotes the $k$-plane passing through $x_0, \ldots, x_k$, and where $C$ depends only on $n$ and $C_\mu$.

**Lemma 5.4.3** (Lemma 7.4 and Remark 7.5 of [To11]). Let $Q \in \mathcal{D}$ and $x_0, \ldots, x_n \in Q$ be like in Lemma 5.4.2. Denote $r = \text{diam}(Q)$, and let $m, p \in \mathbb{Z}$ be such that $t \geq s > 4r$ for $t = 2^{-p}$ and $s = 2^{-m}$. Suppose that $A(x_0, 2^{-m-1/2}, 2^{-m+1/2}) \cap \text{supp}\mu \neq \emptyset$. Then any point $x_{n+1} \in 3Q$ satisfies

$$\text{dist}(x_{n+1}, L_0) \lesssim s \sum_{j=1}^{n+1} \sum_{k=p}^{m} |S_k \mu(x_j) - S_k \mu(x_0)| + \frac{r^2}{s} + \frac{rs}{l}, \quad (5.4.1)$$

where $L_0$ is the $n$-plane passing through $x_0, \ldots, x_n$.

The following proposition is an direct consequence of the techniques used in the last section of [To11]. We give the proof for completeness.

**Proposition 5.4.4.** Given $\epsilon_0 > 0$, there exist $\delta_0 > 0$ and $m_0, k_0 \in \mathbb{N}$ depending on $\epsilon_0$, $n$, and $C_\mu$ such that, for all $i \in \mathbb{Z}$ and all $Q \in \mathcal{D}_i$ with $\beta_{1, \mu}(Q) > \epsilon_0$, there exist $k \in \mathbb{Z}$ with $|k| \leq k_0$ and $P \in \mathcal{D}_{i+k+m_0}$ such that $P \subset 4Q$ and $|S_{i+k} \mu(x)| \geq \delta_0$ for all $x \in P$.

**Proof.** Fix $\epsilon_0 > 0$. Let $Q \in \mathcal{D}_i$ such that $\beta_{1, \mu}(Q) > \epsilon_0$. Take points $x_0, \ldots, x_n$ in $Q$ as in Lemma 5.4.2, denote $r = \text{diam}(Q)$, and let $m \in \mathbb{Z}$ to be fixed below such that $4r < 2^{-m} =: s$ and $A(x_0, 2^{-m-1/2}, 2^{-m+1/2}) \cap \text{supp}\mu \neq \emptyset$ (we assume $\text{diam}(\text{supp}\mu) = \infty$). By Lemma 5.4.3, for $t := 2^{-p} \geq s$ to be fixed below and all $x_{n+1} \in 3Q$,

$$\text{dist}(x_{n+1}, L_0) \lesssim s \sum_{j=1}^{n+1} \sum_{k=p}^{m} |S_k \mu(x_j) - S_k \mu(x_0)| + \frac{r^2}{s} + \frac{rs}{l},$$

where $L_0$ is the $n$-plane passing through $x_0, \ldots, x_n$.\[\]
Thus, Given Definition 5.4.5. Then, by integrating on $x_{n+1} \in 3Q$, for some constant $C_1 > 0$ depending only on $n$ and $C_\mu$

\[
eq 0 < \beta_{1,\mu}(Q) \leq \frac{1}{\ell(Q)^{n}} \int_{3Q} \frac{\text{dist}(x_{n+1}, L_0)}{\ell(Q)} \ d\mu(x_{n+1})
\]

\[
\leq C_1 \left( \frac{s}{r} \sum_{k=p}^{m} \left( \frac{1}{\ell(Q)^{n}} \int_{3Q} |S_{k}\mu(x_{n+1})| \ d\mu(x_{n+1}) + \sum_{j=0}^{n} |S_{k}\mu(x_{j})| \right) + \frac{r}{s} + \frac{s}{t} \right).
\]

Thus,

\[
\frac{r}{s} \left( \frac{\epsilon_0}{C_1} - \frac{r}{s} - \frac{s}{t} \right) \leq \sum_{k=p}^{m} \left( \int_{3Q} \frac{|S_{k}\mu(x_{n+1})|}{\ell(Q)^{n}} \ d\mu(x_{n+1}) + \sum_{j=0}^{n} |S_{k}\mu(x_{j})| \right).
\]

We can easily choose $s$ and $t$ big enough (depending on $r$, $\epsilon_0$, and $C_1$) such that, for some constant $\epsilon_1 > 0$ depending only on $\epsilon_0$, $n$ and $C_\mu$,

\[
0 < \epsilon_1 \leq \sum_{k=p}^{m} \left( \int_{3Q} \frac{|S_{k}\mu(x_{n+1})|}{\ell(Q)^{n}} \ d\mu(x_{n+1}) + \sum_{j=0}^{n} |S_{k}\mu(x_{j})| \right).
\]

Notice that, since $t = 2^{-p}$ and $s = 2^{-m}$ where chosen depending on $r \approx 2^{-i}$, the sum on the right hand side of (5.4.2) has a finite number of terms which only depends on $\epsilon_0$, $n$ and $C_\mu$. Therefore, there exists $k_0 \in \mathbb{N}$ and $C_2 > 0$ depending only on $\epsilon_0$, $n$ and $C_\mu$ such that, for some negative integer $k$ with $|k| \leq k_0$ and some $j = 0, \ldots, n$,

\[
\epsilon_1 \leq C_2 \left( \frac{1}{\ell(Q)^{n}} \int_{3Q} |S_{i+k}\mu| \ d\mu + |S_{i+k}\mu(x)| \right),
\]

which implies that there exists $C_3$ (depending on $C_2$) and $z \in 3Q$ such that $\epsilon_1 \leq C_3|S_{i+k}\mu(z)|$.

Given $x \in \text{supp}\mu$, if $|x - z| \leq 2^{-i-k}$, then

\[
|S_{i+k}\mu(x) - S_{i+k}\mu(z)| \leq \int_{|y - z| \leq 2^{-i-k}} ||\nabla (\varphi_{i+k}K)||_{\infty} |x - z| \ d\mu(y)
\]

\[
\lesssim 2^{(i+k)(n+1)}|x - z| \int_{|y - z| \leq 2^{-i-k}} d\mu(y) \lesssim 2^{i+k}|x - z|.
\]

Hence if $|x - z| \leq C_42^{-i-k}$ with $C_4 > 0$ small enough, we have $C_3|S_{i+k}\mu(x) - S_{i+k}\mu(z)| \leq \epsilon_1/2$, so $\epsilon_1/2 \leq C_3|S_{i+k}\mu(x)|$. Therefore, there exist $m_0 \in \mathbb{N}$ depending on $C_4$ (and thus on $\epsilon_0$, $n$, and $C_\mu$) and $P \in \mathcal{D}_{i+k+m_0}$ such that $\epsilon_1/2 \leq C_3|S_{i+k}\mu(x)|$ for all $x \in P$. We can also assume that $P \subset 4Q$ by taking $C_4$ small enough, and since $|k| \leq k_0$ we have $\ell(P) \approx \ell(Q)$. The proposition follows by setting $\delta_0 := \epsilon_1/(2C_3) > 0$.

**Definition 5.4.5.** Given $\epsilon_0 > 0$, let $\delta_0, m_0 > 0$ be as in Proposition 5.4.4. Set

$$
\mathcal{B} := \{Q \in \mathcal{D} : \beta_{1,\mu}(Q) > \epsilon_0\}, \quad \tilde{\mathcal{B}} := \bigcup_{k \in \mathbb{Z}} \{Q \in \mathcal{D}_{k+m_0} : |S_{k}\mu(x)| \geq \delta_0 \text{ for all } x \in Q\}.
$$

Given $P, R \in \mathcal{D}$ with $P \subset R$, we set $F_{\mathcal{B}}^{P} = \sum_{Q \in \tilde{\mathcal{B}} : P \subset Q} \chi_Q$ and $F_{\mathcal{R}}^{P} = \sum_{Q \in \tilde{\mathcal{B}} : Q \subset R} \chi_Q$. 
Lemma 5.4.6. Let \( \rho > 0 \). Assume that there exists \( C_0 > 0 \) such that, for all \( R \in \mathcal{D} \),
\[
\int_R \left( F^R \right)^{2/\rho} \, d\mu \leq C_0 \mu(R).
\] (5.4.3)

Then, there exists \( C > 0 \) such that \( \sum_{Q \in \mathcal{G} : Q \subset R} \mu(Q) \leq C \mu(R) \) for all \( R \in \mathcal{D} \).

Proof. Let \( M > 1 \) big enough (it will be fixed below). For \( R \in \mathcal{D} \), set
\[
\text{Tree}(R) := \{ Q \in \mathcal{G} : Q \subset R, \chi_Q F^R_Q \leq M \chi_Q \},
\]
\[
\text{Top}_0(R) := \{ P \in \mathcal{G} : P \subset R, \chi_P F^R_P > M \chi_P, \text{ and } \chi_Q F^R_Q \leq M \chi_Q \}
\]
for all \( Q \in \mathcal{G} \) such that \( P \nsubseteq Q \subset R \). For \( m \geq 1 \), set \( \text{Top}_m(R) := \bigcup_{P \in \text{Top}_{m-1}(R)} \text{Top}_0(P) \), and \( \text{Top}(R) := \bigcup_{m \geq 0} \text{Top}_m(P) \). Notice that if \( R \in \mathcal{G} \) then \( R \in \text{Tree}(R) \), because \( M > 1 \). Notice also that
\[
\{ Q \in \mathcal{G} : Q \subset R \} = \text{Tree}(R) \cup \left( \bigcup_{P \in \text{Top}(R)} \text{Tree}(P) \right),
\] (5.4.4)
and the union is disjoint.

Fix \( R \in \mathcal{D} \). Then, by (5.4.4),
\[
\sum_{Q \in \mathcal{G} : Q \subset R} \mu(Q) = \sum_{Q \in \text{Tree}(R)} \mu(Q) + \sum_{P \in \text{Top}(R)} \sum_{Q \in \text{Tree}(P)} \mu(Q)
= \int_R \sum_{Q \in \text{Tree}(R)} \chi_Q \, d\mu + \int_R \sum_{P \in \text{Top}(R)} \sum_{Q \in \text{Tree}(P)} \chi_Q \, d\mu.
\] (5.4.5)

Given \( x \in R \) and \( P \in \mathcal{D} \) such that \( P \subset R \), by the definition of \( \text{Tree}(P) \), \( \sum_{Q \in \text{Tree}(P)} \chi_Q(x) \leq M \chi_P(x) \). Therefore, by (5.4.5),
\[
\sum_{Q \in \mathcal{G} : Q \subset R} \mu(Q) \leq M \mu(R) + \int_R \sum_{P \in \text{Top}(R)} M \chi_P \, d\mu = M \left( \mu(R) + \sum_{m \geq 0} \sum_{P \in \text{Top}_m(R)} \mu(P) \right).
\] (5.4.6)

We are going to prove that, if \( M \) is big enough,
\[
\sum_{P \in \text{Top}_m(R)} \mu(P) \leq 2^{-m-1} \mu(R)
\] (5.4.7)
for all \( m \geq 0 \), and then, by (5.4.6), we will finally obtain
\[
\sum_{Q \in \mathcal{G} : Q \subset R} \mu(Q) \leq M \mu(R) + M \sum_{m \geq 0} 2^{-m-1} \mu(R) \leq 2M \mu(R),
\]
and the lemma will be proven. We verify (5.4.7) by induction on \( m \), so assume first that \( m = 0 \). Notice that, if \( P, P' \in \text{Top}_0(R) \), then \( P \cap P' = \emptyset \) because of the last condition in the
definition of $\text{Top}_0(R)$. Hence, $\sum_{P \in \text{Top}_0(R)} \chi_P = \chi_U$, where we have set $U := \bigcup_{P \in \text{Top}_0(R)} P$. If $x \in U$, there exists $P \in \text{Top}_0(R)$ such that $x \in P$, so $1 = \chi_P(x) < M^{-2/\rho} (F^R_P(x))^{2/\rho} \lesssim M^{-2/\rho} (F^R(x))^{2/\rho}$, and then using (5.4.3) we have

$$\sum_{P \in \text{Top}_0(R)} \mu(P) = \int_R \sum_{P \in \text{Top}_0(R)} \chi_P \, d\mu = \int_R 1 \, d\mu < M^{-2/\rho} \int_R (F^R)^{2/\rho} \, d\mu \leq \frac{C_0}{M^{2/\rho}} \mu(R).$$

This yields (5.4.7) for $m = 0$ by taking $M > (2C_0)^{\rho/2}$.

Now assume that (5.4.7) holds for some $m \geq 0$ and let us verify (5.4.7) for $m + 1$. We have

$$\sum_{P \in \text{Top}_{m+1}(R)} \mu(P) = \sum_{P \in \text{Top}_m(R)} \sum_{Q \in \text{Top}_0(P)} \mu(Q).$$

Similarly to the case $m = 0$, if we set $U := \bigcup_{Q \in \text{Top}_0(P)} Q$, by (5.4.3) we have

$$\sum_{Q \in \text{Top}_0(P)} \mu(Q) = \int_U 1 \, d\mu < M^{-2/\rho} \int_P (F^P)^{2/\rho} \, d\mu \leq \frac{C_0}{M^{2/\rho}} \mu(P) < \frac{1}{2} \mu(P).$$

Therefore, by this last estimate and the induction hypothesis (5.4.7),

$$\sum_{P \in \text{Top}_{m+1}(R)} \mu(P) < \frac{1}{2} \sum_{P \in \text{Top}_m(R)} \mu(P) \leq 2^{-m-2} \mu(R),$$

and the proof of the lemma is finished.

**Lemma 5.4.7.** Assume that, for some $C_1 > 0$, $\sum_{Q \in \mathcal{B} : Q \subset R} \mu(Q) \leq C_1 \mu(R)$ for all $R \in \mathcal{D}$. Then there exists $C_2 > 0$ such that $\sum_{Q \in \mathcal{B} : Q \subset R} \mu(Q) \leq C_2 \mu(R)$ for all $R \in \mathcal{D}$.

**Proof.** Given $Q \in \mathcal{B}$, by Proposition 5.4.4, there exists $P_Q \in \mathcal{D}_{k+m}$ for some $k \in \mathbb{Z}$ such that $P_Q \subset 4Q$, $\mu(P_Q) \geq C_0 \mu(Q)$, and $\vert S_k \mu(x) \vert \geq \delta_0$ for all $x \in P_Q$. Thus, in particular, $P_Q \in \tilde{\mathcal{B}}$ for all $Q \in \mathcal{B}$. Since $P_Q \subset 4Q$ and $\mu(P_Q) \geq C_0 \mu(Q)$ for all $Q \in \mathcal{B}$, given $P \in \tilde{\mathcal{B}}$ there are finitely many cubes $Q \in \mathcal{B}$ such that $P_Q = P$, and the number of such cubes is bounded above by a constant depending only on $n, C_0, \text{ and } C_\mu$. Hence, since $4R$ is contained in the union of a finite number of cubes of $\mathcal{D}$ with side length $\ell(R)$ an this number only depends on $n$,

$$\sum_{Q \in \mathcal{B} : Q \subset R} \mu(Q) \leq C_0^{-1} \sum_{Q \in \mathcal{B} : Q \subset R} \mu(P_Q) \lesssim \sum_{P \in \mathcal{B} : P \subset 4R} \mu(P) \leq C_1 \mu(R)$$

for all $R \in \mathcal{D}$, as desired.

**Theorem 5.4.8.** Let $\rho > 0$. Given an $n$-dimensional AD regular measure $\mu$, if $\mathcal{V}_\rho \circ \mathcal{R}_\mu$ is a bounded operator in $L^2(\mu)$, then $\mu$ is uniformly $n$-rectifiable.
5.4. $L^2$ boundedness of $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ implies that $\mu$ is uniformly rectifiable

**Proof.** It is easy to see that, if $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ is a bounded operator in $L^2(\mu)$, then $R^\mu_\rho$ is also bounded in $L^2(\mu)$ (fix $\epsilon > 0$, and show that $R^\mu_\epsilon$ is bounded in $L^2(\mu)$ and that $R^\mu_\mu \leq \mathcal{V}_\rho \circ \mathcal{R}^\mu + R^\mu_\rho$).

By Theorem 1.2 in [DS2, Part III, Chapter 1], in order to show that a measure $\mu$ like in Theorem 5.4.8 is uniformly $n$-rectifiable, it is enough to show that $\text{supp} \mu$ satisfies the Weak Geometric Lemma, i.e., that for any $\mathcal{L}$

Since the $\mathcal{K}$

Definition 5.0.3 (remember that, now, $\mathcal{K}$

$\rho > 0$, depending on $\epsilon$ such that, for all $R \in \mathcal{D}$,

$$\int_R (F^R)^{2/\rho} d\mu \leq C \mu(R). \tag{5.4.8}$$

Notice that, for $m \in \mathbb{Z}$ and $f \in L^1(\mu)$, $S_m(f \mu) = T^\mu_\varphi \varphi_{2^{-m+1}} f - T^\mu_\varphi \varphi_{2^{-m}} f$, where $T^\mu_\varphi$ is as in Definition 5.0.3 (remember that, now, $K$ denotes the Riesz kernel), thus

$$\sum_{k \in \mathbb{Z}} |S_k(f \mu)(x)|^\rho \leq \left((V^\rho_\rho \circ T^\mu_\varphi) f(x)\right)^\rho. \tag{5.4.9}$$

Since the $\ell^\rho(\mathbb{Z})$-norm is bigger than the $\ell^\rho(\mathbb{Z})$-norm for $\rho < \tilde{\rho}$, if $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ is bounded in $L^2(\mu)$ for some $\rho > 0$ then there exists $\tilde{\rho} \geq 1$ such that $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ is also bounded in $L^2(\mu)$. Since $\varphi_\mathcal{F}(2 \mu t^2)$ is a convex combination of the functions $\chi_{\{s \in \mathcal{F}: s > \epsilon\}}(t)$ for $\epsilon > 0$, using that $\tilde{\rho} \geq 1$ and Minkowski’s integral inequality it is not hard to show that the $L^2(\mu)$ boundedness of $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ implies the $L^2(\mu)$ boundedness of $\mathcal{V}_\rho \circ T^\mu_\varphi$ (see Subsection 5.3.1, or [CJRW1, Lemma 2.4], for a similar argument). Therefore, for any $M > 0$, we have

$$\| (V^\rho_\rho \circ T^\mu_\varphi) \chi_{MR} \|^2_{L^2(\mu)} \leq C \mu(MR) \leq C \mu(R) \text{ for all } R \in \mathcal{D}. \tag{5.4.10}$$

Fix $\epsilon_0 > 0$ and let $\delta_0, m_0 > 0$ be as in Proposition 5.4.4. Let $R \in \mathcal{D}$. Given $x \in R$ and $k \in \mathbb{Z}$, for any $Q \in \mathcal{D}_{k+m_0} \cap \tilde{\mathcal{B}}$ such that $x \in Q \subset R$ we have $|S_k(\mu)(x)| \geq \delta_0$. Notice that, since $Q \in \mathcal{D}_{k+m_0}$ and $Q \subset R$, there exists $M > 1$ depending only on $n$ and $m_0$ such that $\delta_0 \leq |S_k(\mu)(x)| = |S_k(\chi_{MR})(x)|$. Therefore, using (5.4.9) and that for each $k \in \mathbb{Z}$ there is at most one cube $Q \in \mathcal{D}_{k+m_0}$ such that $x \in Q \subset R$,

$$F^R(x) = \sum_{k \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_{k+m_0} \cap \tilde{\mathcal{B}}} \chi_Q(x) \leq \sum_{k \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_{k+m_0} \cap \tilde{\mathcal{B}}} \delta_0^{-\tilde{\rho}} |S_k(x)(\chi_{MR})(x)|^{-\tilde{\rho}} \leq \delta_0^{-\tilde{\rho}} \sum_{k \in \mathbb{Z}} |S_k(x)(\chi_{MR})(x)|^{-\tilde{\rho}} \leq \delta_0^{-\tilde{\rho}} \left((V^\rho_\rho \circ T^\mu_\varphi) \chi_{MR}(x)\right)^{-\tilde{\rho}}, \tag{5.4.11}$$

and then, by (5.4.10),

$$\int_R (F^R)^{2/\rho} d\mu \leq \delta_0^{-2} \int_R (V^\rho_\rho \circ T^\mu_\varphi) \chi_{MR})^2 d\mu \leq \delta_0^{-2} \| (V^\rho_\rho \circ T^\mu_\varphi) \chi_{MR}\|^2_{L^2(\mu)} \leq C \mu(R)$$

for all $R \in \mathcal{D}$. This yields (5.4.8), and the theorem follows. \qed
Remark 5.4.9. Let \( \{r_m\}_{m \in \mathbb{Z}} \subset (0, \infty) \) be a fixed decreasing sequence defining \( \mathcal{O} \). If there exists \( C > 0 \) such that \( C^{-1}r_m \leq r_m - r_{m+1} \leq Cr_m \) for all \( m \in \mathbb{Z} \), then the last inequality in (5.4.11) still holds if we replace \( \mathcal{V}_{\tilde{\rho}} \) by \( \mathcal{O} \) (by taking from the beginning \( \tilde{\rho} = 2 \)). However, we do not know if it holds for any \( \{r_m\}_{m \in \mathbb{Z}} \subset (0, \infty) \).

5.5 Boundedness of \( \mathcal{V}_{\rho} \circ \mathcal{T} \) from \( M(\mathbb{R}^d) \) to \( L^{1,\infty}(\mathcal{H}^n_\Gamma) \)

The proof of Theorem 5.0.12\((b)\) is a nontrivial modification of the proof of [CJRW2, Theorem B] using the Calderón-Zygmund decomposition developed above. We need the following auxiliary lemma (see Lemma 4.1.2):

Lemma 5.5.1. If \( \Gamma \subset \mathbb{R}^d \) is an \( n \)-dimensional Lipschitz graph with slope strictly less than 1, then \( \mathcal{H}^n_\Gamma(A^d(z, a, b)) \lesssim (b-a)^{\alpha-1} \) for all \( 0 < a \leq b \) and \( z \in \Gamma \).

5.5.1 Proof of Theorem 5.0.12\((b)\)

The beginning of the proof is exactly the same as in Theorem 5.0.12\((a)\) (see Subsection 5.1.2), so we keep the same notation. With very similar arguments to the ones used before (5.1.11), to prove Theorem 5.0.12\((b)\), it is enough to show that

\[
\mu\left( \left\{ x \in \mathbb{R}^d \setminus \hat{\Omega} : (\mathcal{V}_{\rho} \circ \mathcal{T})\nu_b(x) > \lambda/2 \right\} \right) \leq \frac{C}{\lambda} \| \nu \|, \tag{5.5.1}
\]

where \( \mu := \mathcal{H}_\Gamma^{n \cap B} \), \( B \subset \mathbb{R}^d \) is some fixed ball, \( \nu \in M(\mathbb{R}^d) \) has compact support, and \( \hat{\Omega} = \bigcup_j 2Q_j \).

Let us prove (5.5.1). Given \( x \in \text{supp} \mu \), let \( \{\epsilon_m\}_{m \in \mathbb{Z}} \) be a decreasing sequence of positive numbers (which depends on \( x \), i.e. \( \epsilon_m \equiv \epsilon_m(x) \)) such that

\[
(\mathcal{V}_{\rho} \circ \mathcal{T})\nu_b(x) \leq 2 \left( \sum_{m \in \mathbb{Z}} |(K\chi_{\epsilon_{m+1}} \ast \nu_b)(x)|^\rho \right)^{1/\rho}. \tag{5.5.2}
\]

If \( R_j \cap A(x, \epsilon_{m+1}, \epsilon_m) = \emptyset \) then \( (K\chi_{\epsilon_{m+1}} \ast \nu_b')(x) = 0 \), so by (5.5.2) and the triangle inequality,

\[
(\mathcal{V}_{\rho} \circ \mathcal{T})\nu_b(x) \leq 2 \left( \sum_{m \in \mathbb{Z}} \left| \sum_{j : R_j \cap A(x, \epsilon_{m+1}, \epsilon_m)} (K\chi_{\epsilon_{m+1}} \ast \nu_b')(x) \right|^\rho \right)^{1/\rho} + 2 \left( \sum_{m \in \mathbb{Z}} \left| \sum_{j : R_j \cap A(x, \epsilon_{m+1}, \epsilon_m) \neq \emptyset} (K\chi_{\epsilon_{m+1}} \ast \nu_b')(x) \right|^\rho \right)^{1/\rho} =: 2(IA(x) + BS(x)),
\]

and then,

\[
\mu\left( \left\{ x \in \mathbb{R}^d \setminus \hat{\Omega} : (\mathcal{V}_{\rho} \circ \mathcal{T})\nu_b(x) > \lambda/2 \right\} \right) \leq \mu\left( \left\{ x \in \mathbb{R}^d \setminus \hat{\Omega} : IA(x) > \lambda/8 \right\} \right) + \mu\left( \left\{ x \in \mathbb{R}^d \setminus \hat{\Omega} : BS(x) > \lambda/8 \right\} \right). \tag{5.5.3}
\]
Let us estimate first $\mu(\{x \in \mathbb{R}^d \setminus \hat{\Omega} : IS(x) > \lambda/8\})$. Since the $\ell^\rho$-norm is not bigger than the $\ell^1$-norm for $\rho \geq 1$,

$$IS(x) \leq \sum_{m \in \mathbb{Z}} \sum_{j: R_j \subset A(x, \varepsilon_{m+1}, \varepsilon_m)} |(K\chi_{\varepsilon_{m+1}}^m * \nu_j^\ell)(x)|$$

$$\leq \sum_{m \in \mathbb{Z}} \sum_{j: R_j \subset A(x, \varepsilon_{m+1}, \varepsilon_m)} \int |\chi_{\varepsilon_{m+1}}^m(x-y)K(x-y)d\nu_j^\ell(y)|$$

$$= \sum_{m \in \mathbb{Z}} \sum_{j: R_j \subset A(x, \varepsilon_{m+1}, \varepsilon_m)} \int |\chi_{\varepsilon_{m+1}}^m(x-y)K(x-y)d\nu_j^\ell(y)|$$

$$\leq \sum_j \chi_{\mathbb{R}^d \setminus R_j}(x) \int |K(x-y)d\nu_j^\ell(y)|.$$ (5.5.4)

Notice that

$$\int_{\mathbb{R}^d \setminus \hat{\Omega}} \chi_{\mathbb{R}^d \setminus R_j}(x) \int K(x-y)d\nu_j^\ell(y)\,d\mu(x) \leq \int_{\mathbb{R}^d \setminus R_j} \int K(x-y)d\nu_j^\ell(y)\,d\mu(x)$$

$$\leq \int_{\mathbb{R}^d \setminus \hat{\Omega}} \int K(x-y)d\nu_j^\ell(y)\,d\mu(x) + \int_{2R_j \setminus R_j} \int K(x-y)d\nu_j^\ell(y)\,d\mu(x).$$ (5.5.5)

On one hand, by (5.1.6) and using the $L^2(\mu)$ boundedness of the maximal operator $T_\mu^\star$ (because $\mu$ is uniformly $n$-rectifiable) and that $\mu(2R_j) \leq C\mu(R_j)$ (because $\frac{1}{2}R_j \cap \text{supp } \mu \neq \emptyset$), we get

$$\int_{2R_j \setminus R_j} \int K(x-y)b_j(y)d\mu(y)\,d\mu(x) \leq \int_{2R_j \setminus R_j} T_\mu^\star b_j d\mu \leq \left(\int_{2R_j} (T_\mu^\star b_j)^2\,d\mu\right)^{1/2} \mu(2R_j)^{1/2}$$

$$\lesssim \|b_j\|_{L^2(\mu)} \mu(2R_j)^{1/2} \lesssim \|b_j\|_{L^\infty(\mu)} \mu(2R_j) \lesssim \|\nu\|(Q_j).$$ (5.5.6)

On the other hand, since $\text{supp } w_j \subset Q_j = \frac{1}{2}R_j$ and $|w_j| \leq 1$, if $x \in 2R_j \setminus R_j$ we have

$$\int |K(x-y)w_j(y)|\,d|\nu|(y) \lesssim |\nu|(Q_j)|x-z_j|^{-n},$$

where $z_j$ denotes the center of $R_j$. Hence, using again that $\mu(2R_j) \leq C\mu(R_j) \leq C\ell(R_j)^n$,

$$\int_{2R_j \setminus R_j} \int K(x-y)w_j(y)\,d\nu(y)\,d\mu(x) \leq \int_{2R_j \setminus R_j} \int |K(x-y)w_j(y)|\,d|\nu|(y)\,d\mu(x)$$

$$\lesssim \|\nu\|(Q_j) \int_{2R_j \setminus R_j} |x-z_j|^{-n}\,d\mu(x) \lesssim \|\nu\|(Q_j)\ell(R_j)^{-n}\mu(2R_j) \lesssim \|\nu\|(Q_j).$$ (5.5.7)

Since $\nu_j^\ell(R_j) = 0$, $\text{supp } \nu_j^\ell \subset R_j$, and $\|\nu_j^\ell\| \lesssim |\nu|(Q_j)$ by (5.1.6),

$$\int_{\mathbb{R}^d \setminus 2R_j} \int K(x-y)d\nu_j^\ell(y)\,d\mu(x) \leq \int_{\mathbb{R}^d \setminus 2R_j} \int_{R_j} |K(x-y) - K(x-z_j)|\,d|\nu_j^\ell|(y)\,d\mu(x)$$

$$\lesssim \int_{\mathbb{R}^d \setminus 2R_j} \int_{R_j} \frac{|y-z_j|}{|x-z_j|^{n+1}}\,d|\nu_j^\ell|(y)\,d\mu(x)$$

$$\lesssim \|\nu_j^\ell\| \int_{\mathbb{R}^d \setminus 2R_j} \frac{\ell(R_j)}{|x-z_j|^{n+1}}\,d\mu(x) \lesssim \|\nu_j^\ell\| \lesssim |\nu|(Q_j).$$
Combining this last estimate with (5.5.6), (5.5.7), and the fact that \( \nu^i_b = w_j \nu - b_j \mu \), we obtain from (5.5.5) that
\[
\int_{\mathbb{R}^d} \chi_{\mathbb{R}^d \setminus R_j}(x) \left| \int K(x - y) \, d\nu^i_b(y) \right| \, d\mu(x) \lesssim |\nu|(Q_j).
\]

Finally, by (5.5.4), we conclude
\[
\mu\left( \left\{ x \in \mathbb{R}^d : IS(x) > \lambda/8 \right\} \right) \leq \frac{8}{\lambda} \int_{\mathbb{R}^d} IS(x) \, d\mu(x)
\leq \frac{8}{\lambda} \sum_j \int_{\mathbb{R}^d \setminus \hat{\Omega}} \chi_{\mathbb{R}^d \setminus R_j}(x) \left| \int K(x - y) \, d\nu^i_b(y) \right| \, d\mu(x) \leq \frac{C}{\lambda} \sum_j |\nu|(Q_j) \leq \frac{C}{\lambda} \| \nu \|.
\]

(5.5.8)

Let us estimate \( \mu\left( \left\{ x \in \mathbb{R}^d : BS(x) > \lambda/8 \right\} \right) \). Recall that \( \epsilon_m \equiv \epsilon_m(x) \). We define
\[
\psi^i_m(x) := \begin{cases} 1 & \text{if } R_j \cap \partial A(x, \epsilon_m(x), \epsilon_m(x)) \neq \emptyset, \quad \text{and} \\ 0 & \text{if not} \end{cases}
\]
\[
\theta_k^j(x) := \begin{cases} 1 & \text{if } R_j \cap \partial A(x, 2^{-k-1}, 2^{-k}) \neq \emptyset \\ 0 & \text{if not} \end{cases}
\]

(5.5.9)

Then, by the triangle inequality, for \( x \in \mathbb{R}^d \setminus \hat{\Omega} \),
\[
BS(x) = \left( \sum_{m \in \mathbb{Z}} \left| \sum_j \psi^i_m(x) (Kx_{\epsilon_{m+1}}^m * \nu^i_b)(x) \right|^\rho \right)^{1/\rho}
\leq \left( \sum_{m \in \mathbb{Z}} \left| \sum_j \chi_{\mathbb{R}^d \setminus 2R_j}(x) \psi^i_m(x) (Kx_{\epsilon_{m+1}}^m * \nu^i_b)(x) \right|^\rho \right)^{1/\rho}
+ \left( \sum_{m \in \mathbb{Z}} \left| \sum_j \chi_{2R_j \setminus 2Q_j}(x) \psi^i_m(x) (Kx_{\epsilon_{m+1}}^m * \nu^i_b)(x) \right|^\rho \right)^{1/\rho}
\leq \left( \sum_{m \in \mathbb{Z}} \left| \sum_j \chi_{\mathbb{R}^d \setminus 2R_j}(x) \psi^i_m(x) (Kx_{\epsilon_{m+1}}^m * \nu^i_b)(x) \right|^\rho \right)^{1/\rho}
+ \sum_j \chi_{2R_j \setminus 2Q_j}(x) \left( \sum_{m \in \mathbb{Z}} |(Kx_{\epsilon_{m+1}}^m * \nu^i_b)(x)|^\rho \right)^{1/\rho}
=: BS_1(x) + BS_2(x).
\]

(5.5.10)

Notice that \( BS_2(x) \leq \sum_j \chi_{2R_j \setminus 2Q_j}(x) (V_{\rho} \circ T) \nu^i_b(x) \). Since \( \rho \geq 1 \), for \( x \in 2R_j \setminus 2Q_j \),
\[
(V_{\rho} \circ T) \nu^i_b(x) \leq (V_{\rho} \circ T)(w_j \nu)(x) + (V_{\rho} \circ T)(b_j \mu)(x)
\leq \sum_{m \in \mathbb{Z}} |(Kx_{\epsilon_{m+1}}^m * (w_j \nu))(x)| + (V_{\rho} \circ T^\mu) b_j(x)
\lesssim |\nu|(Q_j)|x - z_j|^{-n} + (V_{\rho} \circ T^\mu) b_j(x),
\]
where $z_j$ denotes the center of $Q_j$ (and $R_j$). Then, similarly to (5.5.6) and (5.5.7), but using the $L^2(\mu)$ boundedness of $V_\rho \circ T^\mu$ given by Theorem 5.0.11, we have

$$
\mu(\{x \in \mathbb{R}^d \setminus \hat{\Omega} : BS_2(x) > \lambda/16\}) \leq \frac{16}{\lambda} \int_{\mathbb{R}^d \setminus \hat{\Omega}} BS_2 \, d\mu
$$

$$
\leq \frac{16}{\lambda} \sum_j \chi_{2R_j \setminus Q_j}(V_\rho \circ T)\nu_d \, d\mu = \frac{16}{\lambda} \sum_j \int_{2R_j \setminus Q_j} (V_\rho \circ T)\nu_d \, d\mu
$$

$$
\lesssim \frac{1}{\lambda} \sum_j |\nu|(Q_j) \int_{2R_j \setminus Q_j} |x - z_j|^n \, d\mu(x) + \frac{1}{\lambda} \sum_j \int_{2R_j \setminus Q_j} (V_\rho \circ T^\mu)b_j \, d\mu
$$

$$
\lesssim \frac{1}{\lambda} \sum_j |\nu|(Q_j) \int_{2R_j \setminus Q_j} |(x - z_j)^n \, d\mu(x) + \frac{1}{\lambda} \sum_j \int_{2R_j \setminus Q_j} (V_\rho \circ T^\mu)b_j \, d\mu
$$

$$
\lesssim \frac{1}{\lambda} \sum_j |\nu|(Q_j) + \frac{1}{\lambda} \sum_j \|b_j\|_{L^\infty(\mu)}\mu(R_j) \lesssim \frac{1}{\lambda} \sum_j |\nu|(Q_j) \leq \frac{C}{\lambda} \|\nu\|.
$$

Therefore, to show that $\mu(\{x \in \mathbb{R}^d \setminus \hat{\Omega} : BS(x) > \lambda/8\}) \leq C\lambda^{-1}\|\nu\|$, by (5.5.10) and (5.5.11) it is enough to verify that

$$
\mu(\{x \in \mathbb{R}^d \setminus \hat{\Omega} : BS_1(x) > \lambda/16\}) \leq \frac{C}{\lambda} \|\nu\|.
$$

Without loss of generality, we can assume from the beginning that, for a given $x \in \text{supp} \mu$, either $[\epsilon_{m+1}, \epsilon_m) \subset [2^{-k-1}, 2^{-k})$ for some $k \in \mathbb{Z}$, or $[\epsilon_{m+1}, \epsilon_m) = [2^{-i}, 2^{-k})$ for some $i > k$ (see [CJR2, page 2130]). Thus, if we set $I_k := [2^{-k-1}, 2^{-k})$, we can decompose $\mathbb{Z} = S \cup L$, where

$$
L := \{m \in \mathbb{Z} : \epsilon_m = 2^{-k}, \epsilon_{m+1} = 2^{-i} \text{ for } i > k\},
$$

$$
S := \bigcup_{k \in \mathbb{Z}} S_k, \quad S_k := \{m \in \mathbb{Z} : \epsilon_m, \epsilon_{m+1} \in I_k\}.
$$

Then, since $\rho \geq 1$,

$$
BS_1(x) \leq \left( \sum_{m \in L} \left| \sum_j \chi_{\mathbb{R}^d \setminus 2R_j}(x)\psi_j^m(x)(K\chi_{\epsilon_{m+1}}^m \ast \nu_d^j)(x) \right|^\rho \right)^{1/\rho} + \left( \sum_{m \in S} \left| \sum_j \chi_{\mathbb{R}^d \setminus 2R_j}(x)\psi_j^m(x)(K\chi_{\epsilon_{m+1}}^m \ast \nu_d^j)(x) \right|^\rho \right)^{1/\rho} =: BS_L(x) + BS_S(x),
$$

and we have

$$
\mu(\{x \in \mathbb{R}^d \setminus \hat{\Omega} : BS_1(x) > \lambda/16\}) \leq \mu(\{x \in \mathbb{R}^d \setminus \hat{\Omega} : BS_L(x) > \lambda/32\}) + \mu(\{x \in \mathbb{R}^d \setminus \hat{\Omega} : BS_S(x) > \lambda/32\}).
$$

(5.5.12)
We are going to estimate first \( \mu(\{ x \in \mathbb{R}^d \setminus \hat{\Omega} : BS_L(x) > \lambda/32 \}) \). Given \( x \in \mathbb{R}^d \setminus \hat{\Omega} \) (recall the definitions of \( \psi^j_k(x) \) and \( \theta^j_k(x) \) in (5.5.9)), we have

\[
BS_L(x) \leq \sum_{m \in L} \sum_{j} \chi_{\mathbb{R}^d \setminus 2R_j}(x)(\psi^j_m(x))|K \chi_{L_m+1}^e \ast \nu^j_k(x)|
\]

\[
\leq \sum_{k \in Z} \sum_{j} \chi_{\mathbb{R}^d \setminus 2R_j}(x)(\theta^j_k(x))|K \chi_{L_{2-k}+1} \ast \nu^j_k(x)|
\]

\[
\leq \sum_{j} \sum_{k \in Z: 2^{k+1} > \ell(R_j)} \chi_{\mathbb{R}^d \setminus 2R_j}(x)(\theta^j_k(x))|K \chi_{L_{2-k}+1} \ast \nu^j_k(x)|,
\]

where in the second and third inequalities above we used the following facts, respectively:

- Assume \( m \in L, \epsilon_{m+1} = 2^{-i} \) and \( \epsilon_m = 2^{-i+s} \), with \( i \in \mathbb{Z} \) and \( s \in \mathbb{N} \). Given \( j \) such that \( R_j \cap \partial A(x, \epsilon_{m+1}, \epsilon_m) \neq \emptyset \), if \( R_j \cap A(x, 2^{k-1}, 2^{-k}) \neq \emptyset \) for some \( k \in \mathbb{Z} \), then \( R_j \cap \partial A(x, 2^{k-1}, 2^{-k}) \neq \emptyset \).

- For \( x \in \mathbb{R}^d \setminus 2R_j \), if \( 2^{-k+1} \leq \ell(R_j) \) then we have \( \text{supp} \chi_{L_{2-k}+1}(x \cdot) \cap R_j = \emptyset \), so \( (K \chi_{L_{2-k}+1} \ast \nu^j_k)(x) = 0 \).

Therefore, since \( |(K \chi_{L_{2-k}+1} \ast \nu^j_k)(x)| \lesssim 2^{(k+1)n} \| \nu^j_k \| \),

\[
\mu(\{ x \in \mathbb{R}^d \setminus \hat{\Omega} : BS_L(x) > \lambda/32 \}) \leq \frac{32}{\lambda} \int_{\mathbb{R}^d \setminus \hat{\Omega}} BS_L(x) \, d\mu(x)
\]

\[
\leq \frac{32}{\lambda} \sum_j \sum_{k \in Z: 2^{k+1} > \ell(R_j)} \int_{\mathbb{R}^d \setminus 2R_j} \theta^j_k(x)|(K \chi_{L_{2-k}+1} \ast \nu^j_k)(x)| \, d\mu(x)
\]

\[
\lesssim \frac{1}{\lambda} \sum_j \sum_{k \in Z: 2^{k+1} > \ell(R_j)} 2^{(k+1)n} \| \nu^j_k \| \int \theta^j_k(x) \, d\mu(x).
\]  (5.5.13)

Let us check that \( \int \theta^j_k(x) \, d\mu(x) \lesssim \ell(R_j)2^{-k(n-1)} \). Fix \( k \) and \( j \) such that \( 2^{-k+1} \geq \ell(R_j) \), and take \( u \in \frac{1}{2}R_j \cap \text{supp} \mu \) (this \( u \) exists because of (5.1.3)). There exists \( a > 0 \) depending only on \( d \) such that \( \text{supp} \theta^j_k \subset B(u, 2^{-k}a) \); thus, if \( \ell(R_j) \geq 2^{-k}b \) for some small constant \( b > 0 \), \( \int \theta^j_k \, d\mu \leq \mu(B(u, 2^{-k}a)) \lesssim 2^{-kn} \leq b^{-1}(\ell(R_j)2^{-k(n-1)}) \). On the contrary, if \( \ell(R_j) < 2^{-k}b \) and \( b \) is small enough, then

\[
\text{supp} \theta^j_k \subset A(u, 2^{-k} - b' \ell(R_j)), 2^{-k} + b' \ell(R_j)) \cup A(u, 2^{-k-1} - b' \ell(R_j), 2^{-k-1} + b' \ell(R_j))
\]

for some constant \( b' > 0 \) depending on \( b \) and \( d \) such that \( 2^{-k-1} - b' \ell(R_j) > 0 \). In that case, since \( u \in \text{supp} \mu \), \( \int \theta^j_k \, d\mu = \mu(\text{supp} \theta^j_k) \lesssim \ell(R_j)2^{-k(n-1)} \) (because \( \mu(A(u, r, R)) \lesssim (R - r)R^{n-1} \) for all \( 0 < r \leq R \) by Lemma 5.5.1, since \( \Gamma \) has slope smaller than \( 1 \), as desired.
Thus, if we set $P$ and then by Cauchy-Schwarz inequality,
\[
\lambda \mu(BS_C(x) > \lambda/32) \lesssim \frac{1}{\lambda^2} \sum_{j \in \mathbb{Z}} 2^{(k+1)n} \|P\| \ell(R_j) 2^{-(n-1)}
\]
\[
= \frac{1}{\lambda^2} \sum_{j} \|P\| \ell(R_j) \sum_{k \in \mathbb{Z}: 2^{k+1} > \ell(R_j)} 2^{n+k} \lesssim \frac{1}{\lambda} \sum_{j} \|\nu_j\| \lambda \mu(BS_S(x) > \lambda/32) \leq C \lambda^{-1} \|\nu\|.
\]
(5.5.14)

It only remains to show $\mu(\{x \in \mathbb{R}^d \setminus \hat{\Omega} : BS_S(x) > \lambda/32\}) \leq C \lambda^{-1} \|\nu\|$ to finish the proof of the theorem. We set
\[
\Phi_j^m(x) := \chi_{\mathbb{R}^{d} \setminus 2R_j(x)} \psi_m^j(x)(K_{\chi_{\epsilon_{m+1}}}(x) * \nu_j^*(x)).
\]
Recall that $I_r = [2^{-r-1}, 2^{-r})$. Since the $\ell^p$-norm is not bigger than the $\ell^2$-norm,
\[
\mu(\{x \in \mathbb{R}^d \setminus \hat{\Omega} : BS_S(x) > \lambda/32\}) \lesssim \frac{1}{\lambda^2} \sum_{j \in \mathbb{Z}} \left| \sum_{m \in \mathcal{S}} \sum_{k \in \mathbb{Z}} \Phi_j^m(x) \right|^2 d\mu(x)
\]
\[
= \frac{1}{\lambda^2} \sum_{j \in \mathbb{Z}} \left| \sum_{m \in \mathcal{S}} \sum_{r \geq k+1} \nu_j^m(x) \right|^2 d\mu(x)
\]
\[
= \frac{1}{\lambda^2} \sum_{j \in \mathbb{Z}} \left| \sum_{m \in \mathcal{S}} \sum_{r \geq k-1} \nu_j^m(x) \right|^2 d\mu(x),
\]
and then by Cauchy-Schwarz inequality,
\[
\mu(\{x \in \mathbb{R}^d \setminus \hat{\Omega} : BS_S(x) > \lambda/32\}) \lesssim \frac{1}{\lambda^2} \sum_{j \in \mathbb{Z}} \left( \sum_{r \geq k-1} \nu_j^m(x) \right)^2 \left| \sum_{j \in \mathbb{Z}} 2^{(r-k)/2} \sum_{j : \ell(R_j) \in I_r} \Phi_j^m(x) \right|^2 d\mu(x)
\]
\[
\lesssim \frac{1}{\lambda^2} \sum_{j \in \mathbb{Z}} \left( \sum_{r \geq k-1} \nu_j^m(x) \right)^2 \left| \sum_{j \in \mathbb{Z}} 2^{(r-k)/2} \sum_{j : \ell(R_j) \in I_r} \Phi_j^m(x) \right|^2 d\mu(x).
\]
Thus, if we set $P^r_m(x) := \sum_{j : \ell(R_j) \in I_r} \Phi_j^m(x)$, we have seen that
\[
\mu(\{x \in \mathbb{R}^d \setminus \hat{\Omega} : BS_S(x) > \lambda/32\}) \lesssim \frac{1}{\lambda^2} \sum_{j \in \mathbb{Z}} \left( \sum_{r \geq k-1} \nu_j^m(x) \right)^2 \left| P^r_m(x) \right|^2 d\mu(x).
\]
(5.5.15)

Let us estimate $P^r_m(x)$ for $m \in \mathcal{S}_k$ and $r \geq k - 1$. Since $\|\nu_j^m\| \lesssim \|\nu\| (Q_j) \lesssim \|\nu\| (3Q_j) \lesssim \lambda \mu(6Q_j)$ by (5.1.6) and (5.1.3), we have
\[
|P^r_m(x)| \lesssim \sum_{j : \ell(Q_j) \in I_r} \chi_{\mathbb{R}^{d} \setminus 2R_j(x)}(x) |(K_{\chi_{\epsilon_{m+1}}}(x) * \nu_j^*(x))|
\]
\[
\lesssim \sum_{j : \ell(Q_j) \in I_r} \chi_{\mathbb{R}^{d} \setminus 2R_j(x)}(x) \nu_j^m(x) 2^{kn} \|\nu_j^m\| \lesssim \sum_{j : \ell(Q_j) \in I_r} \sum_{6\ell(Q_j) \in I_r, 6Q_j \cap \Delta(x, \epsilon_{m+1}, \epsilon_m) = \emptyset} 2^{kn} \lambda \mu(6Q_j).
\]
(5.5.16)
It is not difficult to see that, if \( \sum_j \chi_{Q_j} \leq C \) for some \( C > 0 \), then \( \sum_{j: 6Q_j \in I_r} \chi_{6Q_j} \leq C' \) for all \( r \in \mathbb{Z} \), where \( C' > 0 \) only depends on \( C \) (that is, the family of cubes \( \mathcal{F} := \{6Q_j\}_{j: 6Q_j \in I_r} \) has finite overlap uniformly in \( r \in \mathbb{Z} \)). We set

\[
\Upsilon := \sum_{j: 6Q_j \in I_r, 6Q_j \cap \partial A(x, \epsilon m, \epsilon m) \neq \emptyset} \chi_{6Q_j}.
\]

If \( 2^{-k}a \leq 2^{-r} \leq 2^{-k+1} \) for some small constant \( a > 0 \) (recall that we are assuming \( r \geq k-1 \)), then there exists a constant \( b > 0 \) depending only on \( d \) and \( a \) such that \( \supp \Upsilon \subset B(x, b2^{-k}) \), and then, by the finite overlap of the family \( \mathcal{F} \),

\[
\sum_{j: 6Q_j \in I_r, 6Q_j \cap \partial A(x, \epsilon m, \epsilon m) \neq \emptyset} \mu(6Q_j) = \int_{B(x, b2^{-k})} \Upsilon \, d\mu \leq C' \mu(B(x, b2^{-k})) \lesssim 2^{-kn} \approx 2^{-r}2^{-k(n-1)}.
\]

On the contrary, if \( 2^{-k}a \geq 2^{-r} \) for a small enough (depending on \( d \)), then there exists a constant \( b > 0 \) depending only on \( d \) and \( a \) such that \( 2^{-k-1} > 2^{-r}b \) and \( \supp \Upsilon \subset A(x, \epsilon m - 2^{-r}b, \epsilon m + 2^{-r}b) \cup A(x, \epsilon m b, \epsilon m + 2^{-r}b) \), and then, since \( m \in \mathcal{S}_k \), \( x \in \supp \mu \) and the slope of \( \Gamma \) is smaller than 1, by Lemma 5.5.1 we have \( \mu(\supp \Upsilon) \leq \mu(A(x, \epsilon m - 2^{-r}b, \epsilon m + 2^{-r}b)) + \mu(A(x, \epsilon m - 2^{-r}b, \epsilon m + 2^{-r}b)) \lesssim 2^{-r}2^{-k(n-1)} \), thus by the finite overlap of the family \( \mathcal{F} \),

\[
\sum_{j: 6Q_j \in I_r, 6Q_j \cap \partial A(x, \epsilon m, \epsilon m) \neq \emptyset} \mu(6Q_j) = \int_{\supp \Upsilon} \Upsilon \, d\mu \lesssim \mu(\supp \Upsilon) \lesssim 2^{-r}2^{-k(n-1)}.
\]

In any case, from (5.5.16) we get \( |P_m^r(x)| \lesssim 2^k \lambda 2^{-r}2^{-k(n-1)} = 2^{-k} \lambda \). Therefore, using (5.5.15) we obtain that

\[
\mu\left\{ x \in \mathbb{R}^d \setminus \hat{\Omega} : BS_{\mathcal{S}}(x) > \lambda / 32 \right\} \lesssim \frac{1}{\lambda} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^d \setminus \hat{\Omega}} \sum_{m \in \mathcal{S}_k, r \geq k-1} 2^{(k-r)/2} |P_m^r(x)| \, d\mu(x)
\]

\[
\leq \frac{1}{\lambda} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^d \setminus \hat{\Omega}} \sum_{m \in \mathcal{S}_k, r \geq k-1} 2^{(k-r)/2} \sum_{j: \ell(R_j) \in I_r, R_j \cap \partial A(x, \epsilon m, \epsilon m) \neq \emptyset} |(K\chi_{\epsilon m+1}^r * \nu^r_0)(x)| \, d\mu(x)
\]

\[
\leq \frac{1}{\lambda} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^d \setminus \hat{\Omega}} \sum_{m \in \mathcal{S}_k, r \geq k-1} 2^{(k-r)/2} \sum_{j: \ell(R_j) \in I_r, R_j \cap A(x, 2^{-k-1}, 2^{-k}) \neq \emptyset} 2^k |\nu^r_0|(A(x, \epsilon m, \epsilon m)) \, d\mu(x)
\]

\[
\leq \frac{1}{\lambda} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^d \setminus \hat{\Omega}} \sum_{m \in \mathcal{S}_k, r \geq k-1} 2^{(k-r)/2+kn} \sum_{j: \ell(R_j) \in I_r, R_j \cap A(x, 2^{-k-1}, 2^{-k}) \neq \emptyset} |\nu^r_0|(A(x, 2^{-k-1}, 2^{-k})) \, d\mu(x).
\]

Hence, if we set

\[
\tau_k^r(x) := \begin{cases} 
1 & \text{if } R_j \cap A(x, 2^{-k-1}, 2^{-k}) \neq \emptyset \\
0 & \text{if not}
\end{cases}
\]
we obtain
\[
\mu(\{ x \in \mathbb{R}^d \setminus \hat{\Omega} : BS_S(x) > \lambda/32 \}) \leq \frac{1}{\lambda} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^d \setminus \hat{\Omega}} \sum_{\substack{r \in \mathbb{Z}: \ r \geq k-1 \ 
 j: \ell(R_j) \in I_r}} 2^{(k-r)/2+kn} \| \nu^j_k \| \tau^j_k(x) \, d\mu(x) \\
= \frac{1}{\lambda} \sum_{k \in \mathbb{Z}} \sum_{\substack{r \in \mathbb{Z}: \ r \geq k-1 \ 
 j: \ell(R_j) \in I_r}} 2^{(k-r)/2+kn} \| \nu^j_k \| \int_{\mathbb{R}^d \setminus \hat{\Omega}} \tau^j_k \, d\mu.
\]

Notice that, if \( \ell(R_j) \in I_r \) and \( r \geq k-1 \), then \( \ell(R_j) < 2^{-k+1} \). Hence, there exists a constant \( C > 0 \) such that \( \text{supp} \tau^j_k \subset B(z_j, C2^{-k}) \) for all \( \ell(R_j) \in I_r \) and all \( r \geq k-1 \) (recall that \( z_j \) is the center of \( R_j \)), and then \( \int_{\mathbb{R}^d \setminus \hat{\Omega}} \tau^j_k \, d\mu \leq \mu(B(z_j, C2^{-k})) \lesssim 2^{-kn} \). Therefore, by exchanging the order of summation and using that \( \| \nu^j_k \| \lesssim |\nu|(Q_j) \), we finally obtain
\[
\mu(\{ x \in \mathbb{R}^d \setminus \hat{\Omega} : BS_S(x) > \lambda/32 \}) \lesssim \frac{1}{\lambda} \sum_{k \in \mathbb{Z}} \sum_{\substack{r \in \mathbb{Z}: \ r \geq k-1 \ 
 j: \ell(R_j) \in I_r}} 2^{(k-r)/2} \| \nu^j_k \| \\
= \frac{1}{\lambda} \sum_{j} |\nu|(Q_j) \sum_{\substack{r \in \mathbb{Z}: \ 2^{-r-1} \leq \ell(R_j) < 2^{-r} \ 
 k \in \mathbb{Z}: \ k \leq r+1}} 2^{(k-r)/2} \quad (5.5.17)
\]
\[
\lesssim \frac{1}{\lambda} \sum_{j} |\nu|(Q_j) \leq \frac{C}{\lambda} \| \nu \|.
\]

The estimate (5.5.1) is a direct consequence of (5.5.3), (5.5.8), (5.5.10), (5.5.11), (5.5.12), (5.5.14), and (5.5.17).
Bibliography


